Exercises for Thermodynamic Geometry Lectures
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1. Solve the non-linear programming problem in K variables ΔL_1, ΔL_1,...,ΔL_K,

$$\min \sum_{i=1}^{K} \Delta L_i^2, \text{ with } \sum_{i=1}^{K} \Delta L_i = L = \text{constant} \tag{1}$$

and evaluate the minimum value of the objective.

2. Given the metric matrix for the ideal gas from Prof. Andresens lecture

$$D^2 S(U, V) = \frac{\partial^2 S}{\partial X_i \partial X_j} = -\begin{pmatrix} C_v/U^2 & 0 \\ 0 & R/V^2 \end{pmatrix} \tag{2}$$

find the length of an isotherm. Assume constant heat capacity and recall that the energy of an ideal gas is a function of temperature alone.

3. In Professor Kellers sandpile, how much work must be wasted if each grain of sand has a mass of 0.1 grams and we start with a one kilo pile of sand sitting on top of a piston with an ideal gas working fluid in contact with a reservoir at temperature T= 298.15 K and pressure p= 1 atm.

4. The two forms of the metric given in Prof. Andresens lecture are supposed to obey the relation $-TD^2S = D^2U$. This is not apparent in the matrices presented (see equations 2 and 3). Find and interpret the matrix V such that $V^T(D^2U)V = -TD^2S$.

$$D^2 U(S, V) = -\begin{pmatrix} T/C_v & -p/C_v \\ -p/C_v & \gamma p/V \end{pmatrix} \tag{3}$$

5. Consider a system with possible states indexed $1, 2, ...M$. Given N (large) copies of the system, the number of ways of achieving occupation numbers n_1, n_2, $...n_m$ is the multinomial coefficient

$$\Omega = \frac{N!}{n_1! n_2! ... n_m!}, \tag{4}$$

show that $k \log \Omega$ equals $-kN \sum_i p_i \log p_i$, where $p_i = n_i/N$. (Use Stirling's approximation $\log(n!) \approx n \log(n) - n$).

6. Show that if

$$p_i = \exp(-E_i/T)/Z,$$

where Z is the partition function

$$Z = \sum_j \exp(-E_j/T,)$$

then

$$dL^2 = \sum_i dp_i^2/p_i = dU D^2 S dU,$$

where $U = \sum_i E_i p_i$ and $S = -k \sum_i p_i \ln(p_i)$

7. Show that with everything as in the previous problem, $dL = dU/\sigma(U)$, where $\sigma(U)$ is the standard deviation of the energy at equilibrium.

8. Show that if we define a new set of coordinates $v_i = \sqrt{p_i}$, then the metric becomes the identity matrix, the simplex $\sum p_i = 1$ becomes the positive orthant of the sphere $\sum v_i^2 = 1$ and the distance becomes the angle between the vectors v.

Open and probably doable problems (dedicated to Manuel Santoro and the Schön-Janssen group):

1. What is the thermodynamic distance between two configurations of a solid with given composition.

2. What is the classical statistical mechanical version of thermodynamic length?