Math 254 Practice Final

1. (10 pts) Find values of a and b such that the polynomial $p(x) = ax^4 + bx$ has a horizontal tangent at the point (1,3).

2. (20 pts) For which values of a and b does the following linear system

$$\begin{aligned} ax_1 + x_2 + x_3 &= 4 \\ x_1 + x_2 &= 6 \\ x_1 + x_3 &= b \end{aligned}$$

a. have one solution?
b. have no solutions?
c. have infinitely many solutions?

3. (10 pts) Find the following determinant

$$\begin{vmatrix} 5 & 31 & 0 & 0 & 8 \\ 2 & 8 & 0 & 0 & 7 \\ 75 & -87 & 1 & 86 & -21 \\ 0 & 1 & 0 & 0 & 0 \\ 38 & -47 & 0 & 1 & 92 \end{vmatrix}$$

4. (20 pts) Find the inverses of the following matrices

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{-1} =$$

5. (20 pts) A linear transformation $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ sends $\mathbf{e}_1$ to $2\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_2$ to $-3\mathbf{e}_1$, and $\mathbf{e}_3$ to $5\mathbf{e}_3$. What is the standard matrix of $\mathbf{T}$? What is the rank of $\mathbf{T}$?

6. (30 pts) Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 12 & 15 \\ 1 & 2 & 4 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 5 \\ 3 & 6 & 10 & 30 & 0 & 0 & 0 \\ 1 & 2 & 3 & 9 & 0 & 0 & 0 \end{pmatrix}$$

a. Find the rank and nullity of A and of A^T.
b. Find bases for the row space and column space of A.
c. Find bases for N=(null space of A) and for $N^\perp$.

7. (20 pts) (a) What is the transition matrix P from the basis

$$B' = \{\mathbf{u}_1' = 2\mathbf{e}_1 - \mathbf{e}_2, \mathbf{u}_2' = -3\mathbf{e}_1\}$$

to the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$?

(b) If $[u]_B = [1,2]$ and $[v]_{B'} = [3,4]$, find $[u]_{B'}$ and $[v]_B$.

8. (20 pts) (a) Find an orthonormal basis for $\mathbb{R}^2$ by applying the Gram-Schmidt orthonormalization process to the basis $\{(3,4),(1,1)\}$.

(b) Find the inner product between the vectors $(3,4)$ and $(1,1)$ for the inner product generated by $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

9. (20 pts) Find the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

and indicate the algebraic and geometric multiplicity for each eigenvalue you found.

10. (30 pts) Find an orthogonal matrix P that diagonalizes $B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{pmatrix}$ and find $P^T BP$.

Part 2: Definitions

Give a short but correct definition of each of the following terms by completing the lines below:

A set S is *linearly independent* iff

A set $S \subset V$ *spans* V iff

A set S is a *basis* of a vector space V iff

The *dimension* of a vector space V is

The *rank* of a matrix A is

The *nullity* of a matrix A is

A number λ is an *eigenvalue* of the linear transformation A iff

A matrix P *diagonalizes* the matrix A iff