5.3	5.4	Total
10	19	29

5.3

1) Explain why the following are linearly dependant sets of vectors. (Solve this problem by inspection.)

(a) $u_1 = (-1, 2, 4)$ and $u_2 = (5, -10, -20)$ in R^3

A set with exactly two vectors is linearly independent if and only if neither vector is scalar multiple of the other.

Here $\boxed{u_2 = -5u_1}$

(d) $A = \begin{bmatrix} -3 & 4 \\ 2 & \emptyset \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & \emptyset \end{bmatrix}$ in M_{22}

Here $\boxed{B = -A}$

2) Which of the following sets of vectors in R^3 are linearly dependant?

(a) $(4, -1, 2)$, $(-4, 10, 2)$

Equation $(4, -1, 2) = k(-4, 10, 2)$ the vectors is not satisfied by any values of k, the vectors

$\boxed{(4,-1,2) \text{ \& } (-4,10,2) \text{ are linearly independant.}}$

(b) $(-3, \emptyset, 4)$, $(5, -1, 2)$ & $(1, 1, 3)$

let $a = (-3, \emptyset, 4)$

$b = (5, -1, 2)$

& $c = (1, 1, 3)$

Then the vector equation $k_1 a + k_2 b + k_3 c = \emptyset$ has at least one solution namely $k_1 = k_2 = k_3 = \emptyset$

If this is the only solution, then the vectors a, b and c form a linear independent set. If there are other solutions, then the vectors form a linear dependent set. In terms of components, the vector equation becomes

$$k_1(-3,0,4)+k_2(5,-1,2)+k_3(1,1,3)=(0,0,0)$$

or equivalently

$$(-3k_1+5k_2+k_3,\ -k_2+k_3,\ 4k_1+2k_2+3k_3)=(0,0,0)$$

$$-3k_1+5k_2+k_3=0$$
$$\cancel{k_2}\ -k_2+k_3=0$$
$$4k_1+2k_2+3k_3=0$$

Thus a, b, c form a linear dependent set if the system has a nontrivial solution, or a linear independent set if it has only the trivial solution.

Since the coefficient matrix

$$\begin{bmatrix}-3 & 5 & 1\\ 0 & -1 & 1\\ 4 & 2 & 3\end{bmatrix} \quad \begin{aligned}&=(-3)(-5)-(5)(-4)+(1)(-4\\ &=15+20+4\\ &=39\neq 0\end{aligned}$$

has a nonzero determinant.

$\therefore$ this system has only the trivial solution

$\boxed{\text{linearly independent}}$

(c) $(8,-1,3)$ & $(4,0,1)$

Equation $(8,-1,3)=k(4,0,1)$

is not satisfied by any values of k.

$\boxed{\text{linearly independent.}}$

(d) $(-2,0,1)$, $(3,2,5)$, $(6,-1,1)$ & $(7,0,-2)$

A set in R^3 with more than three vectors is linearly dependent.

$\therefore$ $\boxed{\text{Linearly dependent.}}$

5) Assume that v_1, v_2 and v_3 are vectors in R^3 that have their initial points at origin. In each part, determine whether the three vectors lie in a plane.

(a) $v_1 = (2, -2, \emptyset)$, $v_2 = (6, 1, 4)$, $v_3 = (2, \emptyset, -4)$

In R^3, a set of three vectors is linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin. Thus if a set of three vectors is linearly dependent then they line in the same plane

So, we will determine whether the vectors form a linearly dependent set or a linearly independent set.

The vector equation

$$k_1v_1 + k_2v_2 + k_3v_3 = \emptyset.$$

has atleast one solution namely $k_1 = k_2 = k_3 = \emptyset$

If this is the only solution, then the vectors v_1, v_2 & v_3 form a linearly independent set. If there are othe solutions, then the vectors form a linearly dependent se

In terms of components, the vector equation becomes

$$k_1(2, -2, \emptyset) + k_2(6, 1, 4) + k_3(2, \emptyset, -4) = \emptyset$$

or, equivalently

$$\begin{aligned} 2k_1 + 6k_2 + 2k_3 &= 0 \\ -2k_1 + k_2 \quad\quad &= \emptyset \\ 4k_2 - 4k_3 &= \emptyset \end{aligned}$$

The augmented matrix for the system is

$$\begin{bmatrix} 2 & 6 & 2 & \emptyset \\ -2 & 1 & \emptyset & \emptyset \\ 0 & 4 & -4 & \emptyset \end{bmatrix}$$

2B

$$\vec{v} \cdot (\vec{v_2} \times \vec{v_3}) = (2,-2,0) \cdot \begin{vmatrix} i & j & k \\ 6 & 1 & 4 \\ 2 & 0 & -4 \end{vmatrix}$$

$$= (2,-2,0) \cdot (-4, 32, -2) = -8 - 64 + 0 = -72 \neq 0$$

$\therefore$ $\boxed{v_1, v_2 \text{ \& } v_3 \text{ do not lie on the same plane.}}$

(b) $v_1 = (-6,7,2)$, $v_2 = (3,2,4)$, $v_3 = (4,-1,2)$

$$k_1(-6,7,2) + k_2(3,2,4) + k_3(4,-1,2) = 0$$

$$\therefore \quad -6k_1 + 3k_2 + 4k_3 = 0$$
$$7k_1 + 2k_2 - k_3 = 0$$
$$2k_1 + 4k_2 + 2k_3 = 0$$

$\therefore$ ~~Augmented matrix~~

$$\cancel{\begin{bmatrix} -6 & 3 & 4 & 0 \\ 7 & 2 & -1 & 0 \\ 2 & 4 & 2 & 0 \end{bmatrix}}$$

if they lie in same plane then $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

$$(-6,7,2) \cdot \begin{vmatrix} i & j & k \\ 3 & 2 & 4 \\ 4 & -1 & 2 \end{vmatrix}$$

$$= (-6,7,2) \cdot (8i + 10j - 11k)$$
$$= (-6,7,2) \cdot (8, 10, -11)$$
$$= -48 + 70 - 22$$
$$= 0$$

$\therefore$ $\boxed{v_1, v_2 \text{ \& } v_3 \text{ lie on the same plane}}$

6) Assume that v_1, v_2 and v_3 are vectors in R^3 that have their initial points at the origin In each part, determine weather the three vectors lie on same plane

(a) $v_1 = (-1, 2, 3)$, $v_2 = (2, -4, -6)$ and $v_3 = (-3, 6, 0)$

suppose that vectors v_1, v_2, v_3 lie on the same line, then there are scalars k_1 and k_2, such that

$$v_2 = k_1 v_1 \quad \text{and} \quad v_3 = k_2 v_1$$

Here

$$v_2 = -2v_1 \quad (k_1 = -2)$$

but $v_3 = k_2 v_1$ is not satisfied by any values of k_2

$\therefore$ Do not lie on the same line.

9) For which real values of λ do the following vectors form a linearly dependent set in R^3?

$v_1 = (\lambda, -1/2, -1/2)$, $v_2 = (-1/2, \lambda, -1/2)$, $v_3 = (-1/2, -1/2, \lambda)$

If v_1, v_2, v_3 are linearly dependent vectors then such k_1, k_2, k_3 exist for which

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ where all of them are not

$$\therefore \quad \begin{aligned} \lambda k_1 - \tfrac{1}{2}k_2 - \tfrac{1}{2}k_3 &= 0 \\ -\tfrac{1}{2}k + \lambda k_2 - \tfrac{1}{2}k_3 &= 0 \\ -\tfrac{1}{2}k - \tfrac{1}{2}k_2 + \lambda k_3 &= 0 \end{aligned}$$

Coefficient matrix of the system is

$$\begin{bmatrix} \lambda & -1/2 & -1/2 \\ -1/2 & \lambda & -1/2 \\ -1/2 & -1/2 & \lambda \end{bmatrix} = 0$$

$$\therefore \quad \lambda\left(\lambda^2 - \frac{1}{4}\right) - \left(-\frac{1}{2}\right)\left(-\frac{\lambda}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(-\frac{\lambda}{2} - \frac{1}{4}\right) = 0$$

$\Rightarrow \frac{1}{4}\lambda^3 - 3\lambda - 1 = 0$

$\therefore (\lambda + 1/2)(8\lambda^2 - 4\lambda - 4) = 0$

$\therefore \boxed{\lambda_1 = -1/2 \quad \text{or} \quad \lambda_2 = 1}$

5.4

2) Which of the following sets of vectors are bases for R^2?

(a) $(2,1)$, $(3,\emptyset)$

They are linearly independent ~~in R^2~~
R^2 is two dimensional.
$\therefore$ $\boxed{\text{Basis for } R^2}$

(b) $(4,1)$, $(-7,-8)$

They are linearly independent.
R^2 is two dimensional.
$\boxed{\text{Basis for } R^2}$

(c) $(\emptyset,\emptyset)$, $(1,3)$

They are linearly dependent.
$\boxed{\text{Not basis for } R^2}$

(d) $(3,9)$, $(-4,-12)$

They are linearly dependent
$\boxed{\text{Not basis for } R^2}$

3) Which of the following sets of vectors are bases for R^3?

(a) $(1,\emptyset,\emptyset)$, $(2,2,\emptyset)$, $(3,3,3)$

$c_1V_1 + c_2V_2 + c_3V_3 = 0$

Here one solution is $c_1 = c_2 = c_3 = \emptyset$
homogeneous system for these vectors

$$c_1 + c_2 + 3c_3 = 0$$
$$2c_2 + 3c_3 = 0$$
$$3c_3 = 0$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} = 6 \neq 0$$

$\Rightarrow$ Linearly independent

$\therefore$ $\boxed{\text{Bases for R3}}$

(b) $(3,1,-4)$, $(2,5,6)$, $(1,4,8)$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

One solution is $c_1 = c_2 = c_3 = 0$

Homogeneous system for these vectors

$$3c_1 + 2c_2 + c_3 = 0$$
$$c_1 + 5c_2 + 4c_3 = 0$$
$$4c_1 + 6c_2 + 8c_3 = 0$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ 4 & 6 & 8 \end{bmatrix} = 3(16) - 2(-8) + 1(-14)$$
$$= 50 \neq 0$$

$\Rightarrow$ Linearly independent.

$\boxed{\text{Bases for R3}}$.

(c) $(2,-3,1)$, $(4,1,1)$, $(0,-7,1)$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

One solution is $c_1 = c_2 = c_3 = 0$

Homogeneous system for these vectors

$$2c_1 + 4c_2 = 0$$
$$-3c_1 + c_2 - 7c_3 = 0$$
$$c_1 + c_2 + c_3 = 0$$

$$A = \begin{bmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{bmatrix} = 2(8) - 4(4) = 0$$

$\Rightarrow$ Linearly dependent.

Even, $2.(2,-3,1) - (4,1,1) = (0,-7,1)$ 4B

$\boxed{\therefore \text{Not Bases for R3}}$

(d) $(1,6,4), (2,4,-1), (-1,2,5)$

$c_1V_1 + c_2V_2 + c_3V_3 = 0$

One solution is $c_1 = c_2 = c_3 = 0$

Homogeneous system for these vectors

$c_1 + 2c_2 - c_3 = 0$

$6c_1 + 4c_2 + 2c_3 = 0$

$4c_1 - c_2 + 5c_3 = 0$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{bmatrix} = 1(22) - 2(22) - 1(-22) = 0$$

Even

$(1,6,4) = (2,4,-1) + (-1,2,5)$

$\therefore$ Linearly dependent

$\therefore$ $\boxed{\text{Not basis for R3}}$

4) Which of the following sets of vectors are bases for P2?

(a) $1 - 3x + 2x^2,\ 1 + x + 4x^2,\ 1 - 7x$

Here $S = \{1, x, x^2\}$

$\therefore$ Given system can also be represented as

$(1,-3,2), (1,1,4), (1,-7,0)$

For $c_1V_1 + c_2V_2 + c_3V_3 = 0$

One solution is $c_1 = c_2 = c_3 = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{bmatrix} = 1(28) - 1(14) + 1(-14) = 0$$

Even $2.(1,-3,2) - (1,1,4) = (1,-7,0)$

$\therefore$ Linearly dependent

$\therefore$ $\boxed{\text{Not bases for P2}}$

(b) $4+6x+x^2$, $-1+4x+2x^2$, $5+2x-x^2$

For $S=\{1, x, x^2\}$ Given system can be represented as

$(4,6,1)$, $(-1,4,2)$; $(5,2,-1)$

But, $(5,2,-1) = (4,6,1) - (-1,4,2)$

$\therefore$ Linearly dependent.

$\boxed{\text{Not basis for P2}}$

(c) $1+x+x^2$, $x+x^2$, x^2

From $S=\{1,x,x^2\}$ Given system can be represented as

$(1,1,1)$, $(0,1,1)$, $(0,0,1)$

For $c_1V_1 + c_2V_2 + c_3V_3 = 0$ one solution is $c_1=c_2=c_3=$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$$

$\therefore$ Linearly independent

$\boxed{\text{Basis for P2}}$

(d) $-4+x+3x^2$, $6+5x+2x^2$, $8+4x+x^2$

From $S=\{1,x,x^2\}$ given system can be represented a

$(-4,1,3)$, $(6,5,2)$ $(8,4,1)$

For $c_1V_1+c_2V_2+c_3V_3=0$ one solution is $c_1=c_2=c_3=$

$$A = \begin{bmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = (-4)(-3) - (6)(-11) + 8(-13)$$

$= -26 \neq 0$

$\Rightarrow$ Linearly independent

$\therefore \boxed{\text{Basis for p2}}$

7) Find co-ordinate vector of w relative to basis

$S = \{u_1, u_2\}$ for R^2

(a) $u_1 = (1,0)$, $u_2 = (0,1)$; $w = (3,-7)$

$w = c_1u_1 + c_2u_2$

$\therefore (3,-7) = c_1(1,0) + c_2(0,1)$

$c_1 = 3$ & $c_2 = -7$

5B

$\therefore \boxed{(W)_S = (3, -7)}$

8) Find the co-ordinate vector of w relative to the basis $S = \{u_1, u_2\}$ of R^2

(a) $u_1 = (1, -1)$, $u_2 = (1, 1)$; $w = (1, 0)$

$$w = c_1 u_1 + c_2 u_2$$

$$(1, 0) = c_1(1, -1) + c_2(1, 1)$$

$$c_1 + c_2 = 1$$

$$-c_1 + c_2 = 0$$

$$\therefore c_1 = c_2 = 1/2$$

$$\boxed{(W)_k = (1/2, 1/2)}$$

9) Find the coordinate vector of v relative to the basis $S = \{v_1, v_2, v_3\}$

(a) $v = (2, -1, 3)$, $v_1 = (1, 0, 0)$, $v_2 = (2, 2, 0)$, $v_3 = (3, 3, 3)$

$$v = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$(2, -1, 3) = k_1(1, 0, 0) + k_2(2, 2, 0) + k_3(3, 3, 3)$$

$$k_1 + 2k_2 + 3k_3 = 2$$

$$2k_2 + 3k_3 = -1$$

$$3k_3 = 3$$

$$\therefore k_3 = 1, \; k_2 = -2, \; k_1 = 3$$

$$\boxed{(v)_S = (3, -2, 1)}$$

10) Find the coordinate vector of p relative to the basis $S = \{p1, p2, p3\}$

(a) $p = 4 - 3x + x^2$, $p1 = 1$, $p2 = x$, $p3 = x^2$

From $S = \{1, x, x^2\}$

$$p = (4, -3, 1)$$

$$p1 = (1, 0, 0)$$

$$p2 = (0, 1, 0)$$

$$p3 = (0, 0, 1)$$

$$p = p1k1 + p2k2 + p3k3$$

$(4,-3,1) = k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1)$

$\therefore k_1 = 4, \quad k_2 = -3, \quad k_3 = 1.$

$$\boxed{(p)_S = (4,-3,1)}$$

11) Find the coordinate vector of A relative to basis $S = \{A_1, A_2, A_3, A_4\}$

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Here S is a basis for the vector space M_{22}

$A = (2, 0, -1, 3)$

$A_1 = (-1, 1, 0, 0)$

$A_2 = (1, 1, 0, 0)$

$A_3 = (0, 0, 1, 0)$

$A_4 = (0, 0, 0, 1)$

We must have scalars k_1, k_2, k_3, k_4 such that

$A = k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4$

$$\therefore \begin{aligned} -k_1 + k_2 &= 2 \\ k_1 + k_2 &= 0 \\ k_3 &= -1 \\ k_4 &= 3 \end{aligned}$$

$\therefore k_1 = -1, \; k_2 = 1, \; k_3 = -1, \; k_4 = 3$

$$\boxed{(A)_S = (-1, 1, -1, 3)}$$

16) Determine the dimension of and a basis for the solution space of the system

$$2x_1 + x_2 + 3x_3 = \emptyset$$
$$x_1 \quad + 5x_3 = \emptyset$$
$$x_2 + x_3 = \emptyset$$

Augmented matrix of the system

$$\begin{bmatrix} 2 & 1 & 3 & \emptyset \\ 1 & \emptyset & 5 & \emptyset \\ \emptyset & 1 & 1 & \emptyset \end{bmatrix}$$

Reducing this matrix to row-echelon form

$$\begin{bmatrix} 1 & 1/2 & 3/2 & \emptyset \\ \emptyset & 1 & 1 & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset \end{bmatrix}$$

General solution of the given system is $x_1 = x_2 = x_3 = \emptyset$

Empty set has zero ~~vector space~~ no vectors and the zero vector space has dimension zero.

$\therefore$ $\boxed{\text{Dimension of the solution plane is zero. Basis is the empty set.}}$

18) Determine bases of the following subspace of R^3

(a) the plane $3x - 2y + 5z = \emptyset$

General solution of the given system is

$x = s, \ y = t \ , z = 0.4t - 0.6s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0.4t - 0.6s \end{bmatrix} = \begin{bmatrix} s \\ \emptyset \\ -0.6s \end{bmatrix} + \begin{bmatrix} \emptyset \\ t \\ 0.4t \end{bmatrix}$$

$$= s\begin{bmatrix}1\\0\\-0.6\end{bmatrix} + t\begin{bmatrix}0\\1\\0.4\end{bmatrix}$$

which shows that the vectors

$$v_1 = \begin{bmatrix}1\\0\\-0.6\end{bmatrix}, \quad v_2 = \begin{bmatrix}\emptyset\\1\\0.4\end{bmatrix}$$

span the solution space

Since they are already independent

$$\boxed{\{(1,0,-3/5), (\emptyset,1,2/5)\} \text{ is the basis for plane } 3x-2y+5z=\emptyset}$$