6.1	6.2	Total
12	10	22

1A

6.1

3) Compute $\langle u,v \rangle$ using the inner product in example 7.

(a)

$$u = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}, \quad v = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

As we know, if

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \text{ and } V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

are any two 2×2 matrices, then the following formula defines an ~~error~~ inner product on M_{22}:

$$\langle U,V \rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Thus,

$$\langle u,v \rangle = 3(-1) + (-2)3 + 4(1) + 8(1)$$
$$= -3 - 6 + 4 + 8$$
$$= 3.$$

$$\boxed{\langle u,v \rangle = 3}$$

7) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. In each part, the given expression is an inner product on R^2. Find a matrix that generates it.

(b) $\langle u,v \rangle = 4u_1v_1 + 6u_2v_2$

As we know, the weighted Euclidean inner product $\langle u,v \rangle = w_1u_1v_1 + w_2u_2v_2$ is the inner product on R^2 generated by

$$A = \begin{bmatrix} \sqrt{w_1} & 0 \\ 0 & \sqrt{w_2} \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

$$\therefore \boxed{A = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}}$$

10) In each part, use the given inner product on R^2 to find

$\|w\|$, where $w = (-1, 3)$.

(a) the Euclidean inner product.

We have

$$
\begin{aligned}
\|w\| &= (w_1, w_2)^{1/2} \\
&= \sqrt{w_1^2 + w_2^2} \\
&= \sqrt{(-1)^2 + (3)^2} \\
&= \sqrt{1 + 9} \\
&= \sqrt{10}
\end{aligned}
$$

$\therefore$ $\boxed{\text{Euclidean inner product} = \sqrt{10}}$

(b) the weighted Euclidean inner product $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

We have

$$
\begin{aligned}
\|w\| &= (w, w)^{1/2} \\
&= \sqrt{3(-1)(-1) + 2(3)(3)} \\
&= \sqrt{3 + 18} \\
&= \sqrt{21}
\end{aligned}
$$

$\boxed{\text{Euclidean inner product} = \sqrt{21}}$

(c) the inner product generated by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

As we know,

$$\|w\| = (w, w)^{1/2} = \sqrt{w^T A^T A w}$$

Since

$$
\begin{aligned}
w^T A^T A w &= [-1\ \ 3] \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= [-1\ \ 3] \begin{bmatrix} 2 & -1 \\ -1 & 13 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= [-5\ \ 40] \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= 5 + 120
\end{aligned}
$$

$= 125$

we have that

$\|W\| = \sqrt{W^T A^T A W} = \sqrt{125} = 5\sqrt{5}$

$\boxed{\text{Euclidean inner product} = \sqrt{125} = 5\sqrt{5}}$

13) Let M_{22} have the inner product in example 7. In each part find $\|A\|$

(a) $A = \begin{bmatrix} -2 & 5 \\ 3 & 6 \end{bmatrix}$

As we know,

$\|A\| = \langle A, A\rangle^{1/2}$

$\|A\| = \sqrt{(-2)^2 + (5)^2 + (3)^2 + (6)^2}$

$= \sqrt{4 + 25 + 9 + 36}$

$= \sqrt{74}$

$\boxed{\|A\| = \sqrt{74}}$

(b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\|A\| = \langle A, A\rangle^{1/2}$

$\|A\| = \sqrt{(0)^2 + (0)^2 + (0)^2 + (0)^2}$

$= 0$

$\boxed{\|A\| = 0}$

14) Let P_2 have the inner product in example 8. Find $d(p, q)$.

$p = 3 - x + x^2$

$q = 2 + 5x^2$

As we know,

$d(p, q) = \|p - q\| = \langle p-q, p-q\rangle^{1/2}$

$= \sqrt{(a_0 - b_0)^2 + (a_1 - b_1)^2 + (a_2 - b_2)^2}$

$= \sqrt{(3-2)^2 + (-1-0)^2 + (1-5)^2}$

$= \sqrt{(1)^2 + (-1)^2 + (-4)^2}$

$= \sqrt{1 + 1 + 16}$

$= \sqrt{18} = 3\sqrt{2}$

$$\boxed{d(p,q) = \sqrt{18} = 3\sqrt{2}}$$

15) let M_{22} have their inner product in example 7.
Find $d(A,B)$.

(a) $A = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$

As we know,

$d(A,B) = \|A-B\| = \langle A-B, A-B\rangle^{1/2}$

$= \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2 + (a_4-b_4)^2}$

$= \sqrt{(2-(-4))^2 + (6-7)^2 + (9-1)^2 + (4-6)^2}$

$= \sqrt{(6)^2 + (-1)^2 + (8)^2 + (-2)^2}$

$= \sqrt{36+1+64+4}$

$= \sqrt{105}$

$$\boxed{d(A,B) = \sqrt{105}}$$

(b) $A = \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 6 & 2 \end{bmatrix}$

As we know,

$d(A,B) = \|A-B\| = \langle A-B, A-B\rangle^{1/2}$

$= \sqrt{(-2-(-5))^2 + (4-1)^2 + (1-6)^2 + (0-2)^2}$

$= \sqrt{(3)^2 + (3)^2 + (-5)^2 + (-2)^2}$

$= \sqrt{9+9+25+4}$

$= \sqrt{47}$

$$\boxed{d(A,B) = \sqrt{47}}$$

16). Suppose that u, v, and w are vectors such that
$\langle u,v\rangle = 2$; $\langle v,w\rangle = -3$; $\langle u,w\rangle = 5$,
$\|u\| = 1$; $\|v\| = -2$, $\|w\| = 7$.

Evaluate the given expression.

(a) $\langle u+v, v+w \rangle$

We have,

$$\langle u+v, v+w \rangle = \langle u, v+w \rangle + \langle v, v+w \rangle$$
$$= \langle u,v \rangle + \langle u,w \rangle + \langle v,v \rangle + \langle v,w \rangle$$
$$= \langle u,v \rangle + \langle v,w \rangle + \langle u,w \rangle + \|v\|^2$$
$$= 2 + (-3) + 5 + (-2)^2$$
$$= 2-3+5+4$$
$$= 8$$

$$\therefore \boxed{\langle u+v, v+w \rangle = 8}$$

(c) $\langle u-v-2w, 4u+v \rangle$

We have,

$$\langle u-v-2w, 4u+v \rangle = \langle u, 4u+v \rangle - \langle v, 4u+v \rangle - 2\langle w, 4u+v \rangle$$
$$= 4\|u\|^2 + \langle u,v \rangle - 4\langle v,u \rangle - \|v\|^2$$
$$- 8\langle w,u \rangle - 2\langle w,v \rangle$$
$$= 4\|u\|^2 - 3\langle u,v \rangle - \|v\|^2 - 8\langle w,u \rangle - 2\langle w,v \rangle$$
$$= 4(1)^2 - 3(2) - (-2)^2 - 8(5) - 2(-3)$$
$$= 4-6-4-40+6$$
$$= -40$$

$$\therefore \boxed{\langle u-v-2w, 4u+v \rangle = -40}$$

6.2

3) Let R^3 have the Euclidean inner product. Let $u = (1,1,-1)$ and $v = (6,7,-15)$. If $\|ku+v\| = 13$. What is k?

Since the norm of the vector is always a non-negative value, then we can rewrite the given expression as,

$$\|ku+v\|^2 = 169.$$

Using the defination of the norm of a vector and the properties of an inner product, we have

$$\|ku+v\|^2 = (ku+v, ku+v) = (ku,ku) + 2(ku,v) + (v,v)$$
$$= k^2(u,u) + 2k(u,v) + (v,v)$$
$$= k^2\|u\|^2 + 2k(u,v) + \|v\|^2$$
$$= k^2((1)^2+(1)^2+(-1)^2) + 2k(1(6)+1(7)-1(-15))$$
$$+ ((6)^2+(7)^2+(-15)^2)$$
$$= k^2(1+1-1) + 2k(6+7+15) + (36+49+225)$$
$$\therefore \|ku+v\|^2 = 3k^2+56k+310$$

So, the given expression can be written as

$$3k^2+56k+310 = 169$$
$$\therefore 3k^2+56k+141 = 0$$

This is a quadratic equation. Solving it for k we get,

$$3k^2+9k+47k+141=0$$
$$3k(k+3)+47(k+3)=0$$
$$\therefore (k+3)(3k+47)=0$$
$$\therefore \boxed{k_1 = -3 \text{ and } k_2 = -47/3}$$

5) Let R^2, R^3, and R^4 have the Euclidean inner product. In each part, find the cosine of the angle between u and v.

(a) $u = (1,-3)$, $v = (2,4)$

The cosine of the angle between nonzero vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ can be found by

$$\cos\theta = \frac{(u,v)}{\|u\|\cdot\|v\|} = \frac{u_1v_1 + u_2v_2}{\sqrt{(u_1)^2+(u_2)^2}\cdot\sqrt{(v_1)^2+(v_2)^2}}$$

$$\therefore \cos\theta = \frac{1(2)-3(4)}{\sqrt{(1)^2+(-3)^2}\cdot\sqrt{(2)^2+(4)^2}}$$
$$= \frac{2-12}{\sqrt{1+9}\cdot\sqrt{4+16}}$$
$$= \frac{-10}{\sqrt{10\times 20}}$$
$$= \frac{-10}{10\sqrt{1\times 2}}$$

$\therefore \cos\theta = \frac{-1}{\sqrt{2}}$

$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$\therefore \boxed{\cos\theta = \frac{-1}{\sqrt{2}} = -0.707}$

(e) $u = (1, 0, 1, 0)$, $v = (-3, -3, -3, -3)$.

$\cos\theta = \frac{\langle u \cdot v \rangle}{\|u\| \cdot \|v\|}$

$\therefore \cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4}{\sqrt{(u_1)^2+(u_2)^2+(u_3)^2+(u_4)^2} \cdot \sqrt{(v_1)^2+(v_2)^2+(v_3)^2+(v_4)^2}}$

$= \frac{1(-3) + 0(-3) + 1(-3) + 0(-3)}{\sqrt{(1)^2+(0)^2+(1)^2+(0)^2} \cdot \sqrt{(-3)^2+(-3)^2+(-3)^2+(-3)^2}}$

$= \frac{-3-3}{\sqrt{1+1} \cdot \sqrt{9+9+9+9}}$

$= \frac{-6}{\sqrt{2 \times 36}}$

$= \frac{-6}{6\sqrt{2}}$

$= \frac{-1}{\sqrt{2}}$

$\therefore \boxed{\cos\theta = -\frac{1}{\sqrt{2}} = -0.707}$

6) Let P_2 have the inner production in example 8 of Section 6.1. Find the cosine of the angle between p and q.

(a) $p = -1 + 5x + 2x^2$, $q = 2 + 4x - 9x^2$

The cosine of the angle between two nonzero vectors $p = a_0 + a_1x + a_2x^2$ and $q = b_0 + b_1x + b_2x^2$ in P_2 can be found by,

$$\cos\theta = \frac{p \cdot q}{\|p\|\|q\|} = \frac{a_0 b_0 + a_1 b_1 + a_2 b_2}{\sqrt{(a_0)^2+(a_1)^2+(a_2)^2} \cdot \sqrt{(b_0)^2+(b_1)^2+(b_2)^2}}$$

thus,

$$\cos\theta = \frac{-1(2) + 5(4) + 2(-9)}{\sqrt{(-1)^2+(5)^2+(2)^2} \cdot \sqrt{(2)^2+(4)^2+(-9)^2}}$$

$$= \frac{-2+20-18}{\sqrt{-1+25+4} \cdot \sqrt{4+16+81}}$$

$$= \frac{0}{\sqrt{28} \cdot \sqrt{101}}$$

$$\therefore \boxed{\cos\theta = 0}$$

8) Let M_{22} have the inner product in example 7 of section 6.1. Find the cosine of the angle between A and B.

(a) $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

The cosine of the angle between two nonzero matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ in M_{22} can be found by

$$\cos\theta = \frac{\langle A, B \rangle}{\|A\|\|B\|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4}{\sqrt{(a_1)^2+(a_2)^2+(a_3)^2+(a_4)^2} \cdot \sqrt{(b_1)^2+(b_2)^2+(b_3)^2+(b_4)^2}}$$

$$\therefore \cos\theta = \frac{2(3) + 6(2) + 1(1) - 3(0)}{\sqrt{(2)^2+(6)^2+(1)^2+(-3)^2} \cdot \sqrt{(3)^2+(2)^2+(1)^2+(0)^2}}$$

$$= \frac{6+12+1-0}{\sqrt{4+36+1+9} \cdot \sqrt{9+4+1}}$$

$$= \frac{19}{\sqrt{50} \cdot \sqrt{14}} = \frac{19}{\sqrt{700}} = \frac{19}{10\sqrt{7}}$$

$$\therefore \boxed{\cos\theta = \frac{19}{10\sqrt{7}} = 0.718}$$

10) Let R^3 have the Euclidean inner product. For which values of k are u and v orthogonal?

(a) $u = (2, 1, 3)$, $V = (1, 7, k)$

We know from the defination that two vectors u and v in an inner product space are called orthogonal if

$$(u, v) = 0.$$

$$\therefore (u, v) = 2(1) + 1(7) + 3(k) = 0$$

$$\therefore 2 + 7 + 3k = 0$$

$$\therefore 3k = -9$$

$$\therefore k = -3.$$

$$\boxed{k = -3}$$

11) Let R^4 have the Euclidean inner product. Find two unit vectors are orthogonal to the three vectors, $u = (2, -1, 4, 0)$, $v = (-1, -1, 2, 2)$, and $w = (3, 2, 5, 4)$.

We'll denote the unit vector orthogonal to u, v and w as $X = (a, b, c, d)$.

Since x is orthogonal to u, v, and w then

$$(x, u) = 0$$
$$(x, v) = 0$$
$$(x, w) = 0$$

$$\therefore \begin{cases} 2a + b - 4c = 0 \\ -a - b + 2c + 2d = 0 \\ 3a + 2b + 5c + 4d = 0 \end{cases}$$

Solving this system we start by multiplying second row by 2 and add it in to the first row,

$$\begin{cases} -b + 4d = 0 \\ -a - b + 2c + 2d = 0 \\ 3a + 2b + 5c + 4d = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -b + 4d = 0 \\ -a - b + 2c + 2d = 0 \\ a + 9c + 8d = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -b + 4d = 0 \\ -a + 2c - 2d = 0 \\ a + 9c + 8d = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -b + 4d = 0 \\ 11c + 6d = 0 \\ a + 9c + 8d = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{-34d}{11} \\ b = 4d \\ c = \frac{-6}{11} d \end{cases}$$

So,

$$x = \left(\frac{-34}{11} d, \ 4d, \ \frac{-6}{11} d, d \right).$$

Since x is a unit vector, then

$$\sqrt{a^2+b^2+c^2+d^2} = 1$$

or

$$a^2+b^2+c^2+d^2 = 1$$

Using the solution of the above system we get

$$\left(\frac{-34d}{11} \right)^2 + (4d)^2 + \left(\frac{-6d}{11} \right)^2 + (d)^2 = 1$$

Solving it for d, we get

$$d = \pm \frac{11}{\sqrt{3249}} = \pm \frac{11}{57}.$$

Thus, the two unit vectors orthogonal to u, v and w are

$$\boxed{\begin{array}{l} x_1 = \left(\frac{-34}{57}, \frac{44}{57}, \frac{-6}{57}, \frac{11}{57} \right) \\ \text{and } x_2 = \left(\frac{34}{57}, \frac{-44}{57}, \frac{6}{57}, \frac{-11}{57} \right) \end{array}}$$

16)

Let $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

(a) Find bases for the row and nullspace of A.

We can find a basis for the row space of A by finding a basis for the row space of any row-echelon form of A. Reducing A to row-echelon form, we obtain

$$R = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By theorem 5.5.6, the nonzero row vectors of R form a basis for the row space of R, and hence form a basis for the row space of A. These basis vectors are

$$r_1 = \begin{bmatrix} 1 & 2 & -1 & 2 \end{bmatrix}$$
$$r_2 = \begin{bmatrix} 0 & 1 & -3 & 2 \end{bmatrix}$$

The null-space of A, is ~~are~~ the solution space of the homogeneous system

$$(1)x_1 + (2)x_2 + (-1)x_3 + (2)x_4 = 0$$
$$(3)x_1 + (5)x_2 + (0)x_3 + (4)x_4 = 0$$
$$(1)x_1 + (1)x_2 + (2)x_3 + (0)x_4 = 0$$

The general solution of the given system is,

$$x_1 = -5s + 2t$$
$$x_2 = 3s - 2t$$
$$x_3 = s$$
$$x_4 = t$$

Therefore, the solution vectors can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5s+2t \\ 3s-2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Thus the vectors

$$v_1 = \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$ form a basis

for the nullspace

$$\boxed{v_1 = \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \cancel{-6}2 \\ -2 \\ 0 \\ 1 \end{bmatrix}}$$

18) Find a basis for the orthogonal complement of the subspace of R^n spanned by the vectors.

(a) $v_1 = (1, -1, 3)$, $v_2 = (5, -4, -4)$, $v_3 = (7, -6, 2)$

Representing in a row space matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

by part(a) of Theorem 6.2.6, the nullspace of A is the orthogonal complement of V.

The nullspace of A is the solution space of the homogeneous system $Ax = 0$. To find the general solution of this system we reduce A to row echelon form

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of the given system is,

$$x_1 = 16s$$
$$x_2 = 11s$$
$$x_3 = s$$

therefore, the solution vector can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16s \\ 19s \\ s \end{bmatrix} = s \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

forms a basis for the nullspace. Expressing this vector in the same notation as v_1, v_2, and v_3, we conclude that the vector

$w_1 = (16, 19, 1)$ forms a basis for the orthogonal complement of the given space

$$\boxed{w = (16, 19, 1)}$$

(c) $v_1 = (1,4,5,2)$, $v_2 = (2,1,3,0)$, $v_3 = (-1,3,2,2)$

Representing in a row-space matrix

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Reducing A to row-echelon form

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution of the given system is

$$x_1 = -s + \tfrac{2}{7}t$$

$$x_2 = -s - \tfrac{4}{7}t$$

$$x_3 = s$$
$$x_4 = t$$

Therefore, the solution vectors can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s + \frac{2}{7}t \\ -s - \frac{4}{7}t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the vectors

$$w_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$$

~~form~~ a basis for the nullspace. Expressing this vectors in the same notation as v_1, v_2 and v_3, we ~~could~~ conclude that the vectors

$w_1 = (-1, -1, 1, 0)$ and $w_2 = (\frac{2}{7}, \frac{-4}{7}, 0, 1)$ form a

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for the orthogonal complement of the given space.

$$\boxed{\begin{aligned} w_1 &= (-1, -1, 1, 0) \\ w_2 &= (2/7, -4/7, 0, 1) \end{aligned}}$$