7.2	7.3	Total
10	07	~~17~~

1A

7.2

1) Let A be a 6×6 matrix with characteristic equation $\lambda^2(\lambda-1)(\lambda-2)^3=0$. What are the possible dimensions for eigenspaces of A?

By defination, if λ_0 is an eigenvalue of an $n\times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0, and the number of times that $\lambda-\lambda_0$ appears as a factor in the characteristic polynomial of A is called the algebric multiplicit. of A.

We know that, if A is a square matrix, then for every eigen value of A, the geomatric multiplicity is less then or equal to the algebric multiplicity.

Let us ~~determine~~ denote the dimension of the eigenspace as D, we also know each eigen value corresponds to at least one eigen vector. Thus in our case we have:

for $\lambda=0$: $D\leq2$
for $\lambda=1$: $D=1$
for $\lambda=2$: $D\leq3$

$$\boxed{\begin{aligned}\lambda=0 &: D\leq2\\ \lambda=1 &: D=1\\ \lambda=2 &: D\leq3\end{aligned}}$$

8) Let

$$A=\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}$$

(a) Find the eigenvalues of A.

The characterestic polynomial of A is:

$$\det(\lambda I-A)=\det\begin{bmatrix}\lambda-4&0&-1\\-2&\lambda-3&-2\\-1&0&\lambda-4\end{bmatrix}$$

$$=(\lambda-3)^2(\lambda-5)$$

$\therefore$ For eigen values,

$\det(\lambda I-A)=0$

$\therefore \ (\lambda-3)^2(\lambda-5) = 0$

$\therefore \ \lambda = 3 \quad \text{OR} \quad \lambda = 5$

$\therefore \ \boxed{\text{Eigen values: } \{3, 5\}}$

(b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

For $\lambda = 3$

$$(3I - A)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Similarly for $\lambda = 5$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$\lambda = 3 \quad \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix} \quad \therefore$ rank of matrix: 1

$\lambda = 5 \quad \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix} \quad \therefore$ rank of matrix : 2

$\boxed{\text{for } \lambda = 3, 5 : \text{rank of matrix} : 1, 2}$

(c) Is A diagonalizable? Justify your conclusion

Yes ~~no~~, because the matrix does ~~not~~ have n-distinct eigenvalues.

$\boxed{\text{~~Yes~~ diagonalizable.}}$

2A

8) Find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+14 & -12 \\ 20 & \lambda-17 \end{bmatrix}$$

$$= (\lambda+14)(\lambda-17) + 240$$

$$= \lambda^2 - 17\lambda + 14\lambda - 238 + 240$$

$$= \lambda^2 - 3\lambda + 2$$

Solving sol^n

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$\Rightarrow \lambda(\lambda-2) - 1(\lambda-2) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2) = 0$$

Yields the following eigenvalues and corresponding eigenvectors

$\lambda = 1, \quad p_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad \lambda = 2 : \quad p_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

There are two basis vectors in total, so matrix A is diagnolizable and

$$P = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$ diagonalizes A.

We know that in this case the matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \dots, \lambda_n$ as its successive diagonal enteries, where λ_n is the eigenvalue corresponding to p_i, for $i = 1, 2, \dots, n$. Thus

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \boxed{P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}.$$

10) Find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

let us find the eigenvalues of the given matrix first. The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{bmatrix} = (\lambda-1)\left[(\lambda-1)^2 - (1)^2\right]$$

$$= (\lambda-1)(\lambda^2 - 2\lambda)$$

Solving this equation

$$(\lambda-1)(\lambda^2-2\lambda) = 0$$

yields the following eigen values and corresponding eigenvector

$$\lambda = 0 : p_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda = 1 : p_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 2 : p_3 : \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so matrix A is diagonalizable and

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$ diagonalizes A.

We know that in this case the matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal enteries, where λ_n is the eigenvalue correspond to p_i, $i = 1, 2, \ldots n$. Thus, we obtain

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

15) Find the geometric and algebric multiplicity of each eigen value, and determine whether A is diagonalizable. If so, find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

The given matrix is a lower triangular matrix. Thus by inspection we find the eigen values:

$\lambda = 0$. Algebric multiplicity equals 2

$\lambda = 1$. Algebric multiplicity equals 1.

Let us find the eigenvectors and thus determine the geometric multiplicities of the eigenvalues. We have

$$\lambda = 0: \; P_1 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \; P_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = 1: \; P_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus we see that for every eigenvalue the geometric multiplicity is equal to the algebric multiplicity. So, A is diagonalizable and

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$ diagonalizes A.

We know that in this case matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \dots, \lambda_n$ as its successive diagonal enteries, where λ_n is the eigenvalue corresponding to P_i for $i = 1, 2, \dots, n$. Thus

Thus, we obtain

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} P &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \\ P^{-1}AP &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}}$$

17) Find the geometric and algebric multiplicity of each eigen value, and determine whether A is diagonalizable. If so, find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The given matrix is an upper triangular matrix. Thus, by inspection we find the eigen values:

$\lambda = -2$: Algebric multiplicity equals 2.

$\lambda = 3$: Algebric multiplicity equals 2.

Let us find the eigen vectors and determine the geometric multiplicities of the eigen values. We have:

$$\lambda = -2: \ p_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ p_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \lambda = 3: \ p_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \ p_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus we see that for every eigenvalue the geometric multiplicity is equal to the algebric multiplicity. So A is diagonalizable and

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

diagonalizes A.
We know that in this case the matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \dots, \lambda_n$ as it's successive diagonal enteries, where λ_n is the eigenvalue corresponding to p_i for $i = 1, 2, \dots, n$.
Thus we obtain

$$P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$$

18) Use the method of example 6 to compute A^{10}, where

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

We know that

$$A^k = PD^kP^{-1}$$

where D is a diagonal matrix.
Let us find the diagonal matrix.
The given matrix is lower diagonal matrix. Thus by inspection we find the eigenvalues and the corresponding eigenvectors.

$$\lambda = 1 : p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} ; \quad \lambda = 2 : p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

There are two basis vectors in total : so matrix A is diagonalizable and

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ diagonalizes } A.$$

We know that in this case the matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries where λ_n is the eigen value corresponding to p_i for $i = 1, 2, \ldots, n$.

Thus we have:

$$D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

So we obtain

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1024 & 2048 \end{bmatrix}$$

$$\boxed{A^{10} = \begin{bmatrix} 1 & 0 \\ -1023 & 2048 \end{bmatrix}}$$

20) Compute stated power.

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) A^{-2301}

We know that

$$A^k = PD^kP^{-1}$$

Where D is a diagonal matrix and k is a positive integer. In our case k is negative. We can write

$$A^{-2301} = (A^{-1})^{2301}.$$

Now we can use the formula stated above for

$$B = A^{-1} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let us find the diagonal matrix.

The given matrix is an upper triangular matrix. Thus by inspection we find the eigenvalues and the corresponding eigen vectors.

$$\lambda = -1 ; \quad p_1 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}, \; p_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \lambda = 1 : \quad p_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So, the matrix B is diagonalizable and

$$P = \begin{bmatrix} -4 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

diagonalizes B

We know that in this case the matrix $P^{-1}BP$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal enteries, where λ_n is the eigenvalue corresponding to p_i, for $i = 1, 2, \ldots, n$

So we obtain

$$B^{2301} = PD^{2301}P^{-1} = \begin{bmatrix} -4 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{2301} & 0 & 0 \\ 0 & (-1)^{2301} & 0 \\ 0 & 0 & 1^{2301} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus,

$$A^{-2301} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\boxed{A^{-2301} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

7.3

1) Find the characteristic equation of the given symmetric matrix, and then by inspection determine the dimensions of the eigenspaces:

(9)
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

The characteristic equation of matrix A is

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{bmatrix} = (\lambda-1)(\lambda-4)-4$$

$$\Rightarrow \quad \lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\Rightarrow \quad \lambda(\lambda-5) = 0$$

Thus the eigen values of A are $\lambda_1 = 0$ and $\lambda_2 = 5$. So, there are 2 eigen spaces of A.

As we know, the dimension of an eigenspace is the nullity $(\lambda I - A)$ corresponding to λ.

So, if $\underline{\lambda = 0}$ then

$$(\lambda I - A) = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}$$

Reducing to row-echelon form has the form $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Thus, the dimension of the eigenspace of A corresponding to $\lambda = 0$ is equal to

nullity $(\lambda I - A) = 2 - \text{rank}(\lambda I - A) = 2 - 1 = 1$.

Analogously, for $\lambda = 5$ we have that

$$(\lambda I - A) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

reducing to row echelon form has the form $\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$

Thus the dimension of the eigenspace of A corresponding to $\lambda = 5$ is equal to

nullity $(\lambda I - A) = 2 - \text{rank}(\lambda I - A) = 2 - 1 = 1$

Dimension of eigenspace corresponding to $\lambda_1 = 0$: 1	
Dimension of eigenspace corresponding to $\lambda_2 = 5$: 1	

(f) $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

The characterestic equation of matrix A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-2 & 1 \\ 0 & 0 & 1 & \lambda-2 \end{bmatrix} = \left(\det \begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix}\right)^2$$

$$= \left((\lambda-2)^2 - 1\right)^2 = (\lambda-3)^2(\lambda-1)^2 = 0.$$

Thus the eigen values of A are $\lambda_{1,2} = 3$ and $\lambda_{3,4} = 1$.

So there are two eigen spaces of A.

As we know, the dimension of an eigenspace is nullity $(\lambda I - A)$ corresponding to λ. ~~to~~

So, if $\lambda = 3$, then

$$(\lambda I - A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

reducing to row echelon form $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, the dimension of eigenspace of A corresponding to $\lambda = 3$ equals to

nullity $(\lambda I - A) = 4 - \text{rank}(\lambda I - A) = 4 - 2 = 2$.

Analogouly, for $\lambda = 1$ we have that

$$(\lambda I - A) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

reducing to row-echelon form has the form

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the dimension of the eigenspace of A corresponding to $\lambda = 1$ equals to
nullity $(\lambda I - A) = 4 - \text{rank}(\lambda I - A) = 4 - 2 = 2$.

> Characteristic equation: $(\lambda-3)^2(\lambda-1)^2 = 0$.
> The dimension of the eigenspace of A corresponding to $\lambda_{1,2}: 3 : 2$
> The dimension of the eigenspace of A corresponding to $\lambda_{3,4}: 1 : 2$

2) Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The characteristic equation of matrix A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-3 & -1 \\ -1 & \lambda-3 \end{bmatrix} = (\lambda-3)^2 - 1$$

$$= (\lambda-2)(\lambda-4) = 0.$$

And we find the following bases for the eigenspace:

$\lambda = 2$: $P_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad \lambda = 4$: $P_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

There are two basis vectors in total, so matrix A is diagonalizable and

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

diagonalizes A. As a check we should verify

$$P^{-1}AP = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

> $P: \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. $\quad P^{-1}AP: \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

4)
$$A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$$

The characteristic equation of matrix A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 6 & 2 \\ 2 & \lambda - 3 \end{bmatrix} = (\lambda-6)(\lambda-3) - 4$$

$$= (\lambda - 2)(\lambda - 7) = 0$$

and we find the following bases for the eigenspaces:

$\lambda = 2$: $p_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, $\lambda = 7$: $p_2 : \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

There are two basis vectors in total, so matrix A is diagonalizable and

$$P = \begin{bmatrix} 1/2 & -2 \\ 1 & 1 \end{bmatrix}$$ diagonalizes A. As a check we should verify that

$$P^{-1}AP = \begin{bmatrix} 2/5 & 4/5 \\ -2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\boxed{\begin{array}{l} P = \begin{bmatrix} 1/2 & -2 \\ 1 & 1 \end{bmatrix} \\ P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 6 & 7 \end{bmatrix} \end{array}}$$

5)
$$A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$$

The characteristic equation of matrix A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda + 2 & 0 & 36 \\ 0 & \lambda + 3 & 0 \\ 36 & 0 & \lambda + 23 \end{bmatrix}$$

$$= (\lambda+2)(\lambda+3)(\lambda+23) + 36(-36)(\lambda+3)$$

$$= (\lambda+3)(\lambda-25)(\lambda+50) = 0$$

and we find the following orthonormal bases for the eigenspaces

7B

$\lambda = -3 : p_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = 25 : p_2 = \begin{bmatrix} -4/5 \\ 0 \\ 3/5 \end{bmatrix}$, $\lambda = -50 : p_3 = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}$

There are two basis vectors in total, so matrix A is diagonalizable and

$$P = \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$ diagonalizes A.

As a check we should verify that

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} \quad P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix}}$$

10) Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

The characteristic equation of this matrix is

$$(\lambda - a + b)(\lambda - a - b) = 0$$

and we find the following orthonormal bases for the eigenspaces:

$\lambda = a - b$: $p_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$; $\lambda = a + b$ $p_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

There are two basis vectors in total, so matrix A is diagonalizable and

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$ diagonalizes A.

$$\boxed{P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

12) (b) Find a matrix P that orthogonally diagonalizes $I - vv^T$ if

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We have that

$$A = I - vv^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The characteristic equation of this matrix is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 1 \\ 0 & \lambda-1 & 0 \\ 1 & 0 & \lambda \end{bmatrix} = (\lambda - 1)^2(\lambda + 1) = 0$$

Thus, the eigenvalues of A are $\lambda_{1,2} = 1$ and $\lambda_3 = -1$.
It can be shown that

$$u_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

form an orthonormal basis for the eigenspace corresponding to $\lambda = 1$.
The eigenspace corresponding to $\lambda = -1$ has

$$u_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$ as an orthonormal basis.

Finally using u_1, u_2 and u_3 as column vector, we obtain

$$P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

which orthogonally diagonalizes A.

$$\boxed{P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}}$$