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FOUNDATIONS OF ANALYSIS

THE ARITHMETIC OF WHOLE, RATIONAL, IRRATIONAL AND COMPLEX NUMBERS

A Supplement to Text-Books on the Differential and Integral Calculus

BY
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PREFACE FOR THE STUDENT

1. Please don't read the Preface for the Teacher.

2. I will ask of you only the ability to read English and to think logically—no high school mathematics, and certainly no advanced mathematics.

To prevent arguments: A number, **no** number, **two** cases, **all** objects of a given totality, etc. are completely unambiguous phrases. "Theorem 1," "Theorem 2," . . . . "Theorem 301" (and the like in the case of axioms, definitions, chapters, and sections) and also "1)", "2)" (used for distinguishing cases) are simply labels for distinguishing the various theorems, axioms, definitions, chapters, sections, and cases, and are more convenient for purposes of reference than if I were to speak, say, of "Theorem Light Blue," "Theorem Dark Blue," and so on. Up to "301," as a matter of fact, there would be difficulty whatever in introducing the so-called positive integers. The first difficulty—overcome in Chapter I—lies in the *totality* of the positive integers

$$1, \ldots$$

with the mysterious series of dots after the comma (in Chapter I, they are called natural numbers), in defining the arithmetical operations upon these numbers, and in the proofs of the pertinent theorems.

I develop corresponding material in each of the chapters in turn: in Chapter 1, for the natural numbers; in Chapter 2, for the positive fractions and positive rational numbers; in Chapter 3, for the positive (rational and irrational) numbers; in Chapter 4, for the real numbers (positive, negative, and zero); and in Chapter 5, for the complex numbers; thus, I speak only of such numbers as you have already dealt with in high school.

In this connection:

3. Please forget what you have learned in school; you haven't learned it.

Please keep in mind everywhere the corresponding portions of your school work; you haven't actually forgotten them.

4. The multiplication table is not to be found in this book, not even the theorem

$$2 \cdot 2 = 4;$$

but I would recommend, as an exercise in connection with Chapter 1, § 4, that you make the following definitions:

$$2 = 1 + 1,$$
$$4 = (((1 + 1) + 1) + 1),$$

and then prove the theorem.

5. Forgive me for "theeing" and "thouing" you.* One reason for my doing so is that this book is written partly *in usum delphinarum*:† for, as is well known (cf. E. Landau *Vorlesungen über Zahlentheorie*, Vol. I, p. V), my daughters have been studying (Chemistry) at the University for several semesters already and think that they have learned the differential and integral calculus in College; and yet they still don't know why

$$x \cdot y = y \cdot x.$$

Berlin, December 28, 1929.

Edmund Landau

* In the original German, Professor Landau uses the familiar "du" [thou] throughout this preface. [*Trans.*]

† For Delphin use. The Delphin classics were prepared by great French scholars for the use of the Dauphin of France, son of King Louis XIV. [*Trans.*]

PREFACE FOR THE TEACHER

This little book is a concession to those of my colleagues (u
fortunately in the majority) who do **not** share my point of vie
on the following question.

While a rigorous and complete exposition of elementary math
matics can not, of course, be expected in the high schools, t
mathematical courses in colleges and universities should acquai
the student not only with the subject matter and results of math
matics, but also with its methods of proof. Even one who studi
mathematics mainly for its applications to physics and to oth
sciences, and who must therefore often discover auxiliary math
matical theorems for himself, can not continue to take ste
securely along the path he has chosen unless he has learned ho
to walk—that is, unless he is able to distinguish between true a
false, between supposition and proof (or, as some say so nicel
between non-rigorous and rigorous proof).

I therefore think it right—as do some of my teachers and c
leagues, some authors whose writings I have found of help, a
most of my students—that even in his first semester the stude
should learn what the basic facts are, accepted as axioms, fro
which mathematical analysis is developed, and how one can pr
ceed with this development. As is well known, these axioms c
be selected in various ways; so that I do not declare it to be i
correct, but only to be almost diametrically opposite to my poi
of view, if one postulates as axioms for real numbers many of t
usual rules of arithmetic and the main theorem of this bo
(Theorem 205, Dedekind's Theorem). I do not, to be sure, pro
the consistency of the five Peano axioms (because that can not
done), but each of them is obviously independent of the precedi
ones. On the other hand, were we to adopt a large number of axion
as mentioned above, the question would immediately occur to t
student whether some of them could not be proved (a shrewd o
would add: or disproved) by means of the rest of them. Since it h
been known for many decades that all these additional axioms c

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CHAPTER V

COMPLEX NUMBERS

CHAPTER I

NATURAL NUMBERS

§ 1

Axioms

We assume the following to be given:

A set (i.e. totality) of objects called natural numbers, possessing the properties—called axioms—to be listed below.

Before formulating the axioms we make some remarks about the symbols $=$ and $\neq$ which will be used.

Unless otherwise specified, small italic letters will stand for natural numbers throughout this book.

If x is given and y is given, then

either x and y are the same number; this may be written

$$x = y$$

($=$ to be read "equals");

or x and y are not the same number; this may be written

$$x \neq y$$

($\neq$ to be read "is not equal to").

Accordingly, the following are true on purely logical grounds:

1) $$x = x$$

for every x.

2) If

$$x = y$$

then

$$y = x.$$

3) If

$$x = y, \quad y = z$$

then

$$x = z.$$

Thus a statement such as

$$a = b = c = d,$$

which on the face of it means merely that

$$a = b, \quad b = c, \quad c = d,$$

contains the additional information that, say,

$$a = c, \quad a = d, \quad b = d.$$

(Similarly in the later chapters.)

Now, we assume that the set of all natural numbers has the following properties:

Axiom 1: *1 is a natural number.*

That is, our set is not empty; it contains an object called 1 (read "one").

Axiom 2: *For each x there exists exactly one natural number, called the successor of x, which will be denoted by x'.*

In the case of complicated natural numbers x, we will enclose in parentheses the number whose successor is to be written down, since otherwise ambiguities might arise. We will do the same, throughout this book, in the case of $x + y$, xy, $x - y$, $-x$, x^y, etc.

Thus, if

$$x = y$$

then

$$x' = y'.$$

Axiom 3: *We always have*

$$x' \neq 1.$$

That is, there exists no number whose successor is 1.

Axiom 4: *If*

$$x' = y'$$

then

$$x = y.$$

That is, for any given number there exists either no number or exactly one number whose successor is the given number.

Axiom 5 (Axiom of Induction): *Let there be given a set $\mathfrak{M}$ of natural numbers, with the following properties:*

I) *1 belongs to $\mathfrak{M}$.*

II) *If x belongs to $\mathfrak{M}$ then so does x'.*

Then $\mathfrak{M}$ contains all the natural numbers.

§ 2

Addition

Theorem 1: *If*

$$x \neq y$$

then

$$x' \neq y'.$$

Proof: Otherwise, we would have

$$x' = y'$$

and hence, by Axiom 4,

$$x = y.$$

Theorem 2: $$x' \neq x.$$

Proof: Let $\mathfrak{M}$ be the set of all x for which this holds true.

I) By Axiom 1 and Axiom 3,

$$1' \neq 1;$$

therefore 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x' \neq x,$$

and hence by Theorem 1,

$$(x')' \neq x',$$

so that x' belongs to $\mathfrak{M}$.

By Axiom 5, $\mathfrak{M}$ therefore contains all the natural numbers, i.e. we have for each x that

$$x' \neq x.$$

Theorem 3: *If*

$$x \neq 1,$$

then there exists one (hence, by Axiom 4, exactly one) *u such that*

$$x = u'.$$

Proof: Let $\mathfrak{M}$ be the set consisting of the number 1 and of all those x for which there exists such a u. (For any such x, we have of necessity that

$$x \neq 1$$

by Axiom 3.)

I) 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then, with u denoting the number x, we have

$$x' = u',$$

so that x' belongs to $\mathfrak{M}$.

By Axiom 5, $\mathfrak{M}$ therefore contains all the natural numbers; thus for each

$$x \neq 1$$

there exists a u such that

$$x = u'.$$

Theorem 4, and at the same time **Definition 1:** *To every pair of numbers x, y, we may assign in exactly one way a natural number, called $x + y$* (+ to be read "plus"), *such that*

1) $x + 1 = x'$ *for every x,*

2) $x + y' = (x + y)'$ *for every x and every y.*

$x + y$ is called the sum of x and y, or the number obtained by addition of y to x.

Proof: A) First we will show that for each fixed x there is at most one possibility of defining $x + y$ for all y in such a way that

$$x + 1 = x'$$

and

$$x + y' = (x + y)' \quad \text{for every } y.$$

Let a_y and b_y be defined for all y and be such that

$$a_1 = x', \quad b_1 = x',$$
$$a_{y'} = (a_y)', \quad b_{y'} = (b_y)' \quad \text{for every } y.$$

Let $\mathfrak{M}$ be the set of all y for which

$$a_y = b_y.$$

I) $$a_1 = x' = b_1;$$

hence 1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$a_y = b_y,$$

hence by Axiom 2,

$$(a_y)' = (b_y)',$$

therefore

$$a_{y'} = (a_y)' = (b_y)' = b_{y'},$$

so that y' belongs to $\mathfrak{M}$.

Hence $\mathfrak{M}$ is the set of all natural numbers; i.e. for every y we have

$$a_y = b_y.$$

B) Now we will show that for each x it is actually possible to define $x + y$ for all y in such a way that

$$x + 1 = x'$$

and

$$x + y' = (x + y)' \quad \text{for every } y.$$

Let $\mathfrak{M}$ be the set of all x for which this is possible (in exactly one way, by A)).

I) For

$$x = 1,$$

the number

$$x + y = y'$$

is as required, since

$$x + 1 = 1' = x',$$
$$x + y' = (y')' = (x + y)'.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let x belong to $\mathfrak{M}$, so that there exists an $x + y$ for all y. Then the number

$$x' + y = (x + y)'$$

is the required number for x', since

$$x' + 1 = (x+1)' = (x')'$$

and

$$x' + y' = (x + y')' = ((x+y)')' = (x' + y)'.$$

Hence x' belongs to $\mathfrak{M}$.

Therefore $\mathfrak{M}$ contains all x.

Theorem 5 (Associative Law of Addition):

$$(x + y) + z = x + (y + z).$$

Proof: Fix x and y, and denote by $\mathfrak{M}$ the set of all z for which the assertion of the theorem holds.

I) $(x + y) + 1 = (x + y)' = x + y' = x + (y + 1)$;

thus 1 belongs to $\mathfrak{M}$.

II) Let z belong to $\mathfrak{M}$. Then

$$(x + y) + z = x + (y + z),$$

hence

$$(x+y)+z' = ((x+y)+z)' = (x+(y+z))' = x+(y+z)' = x+(y+z'),$$

so that z' belongs to $\mathfrak{M}$.

Therefore the assertion holds for all z.

Theorem 6 (Commutative Law of Addition):

$$x + y = y + x.$$

Proof: Fix y, and let $\mathfrak{M}$ be the set of all x for which the assertion holds.

I) We have

$$y + 1 = y',$$

and furthermore, by the construction in the proof of Theorem 4,

$$1 + y = y',$$

so that

$$1 + y = y + 1$$

and 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x + y = y + x,$$

therefore

$$(x + y)' = (y + x)' = y + x'.$$

By the construction in the proof of Theorem 4, we have

$$x' + y = (x + y)',$$

hence

$$x' + y = y + x',$$

so that x' belongs to $\mathfrak{M}$.

The assertion therefore holds for all x.

Theorem 7: $y \neq x + y.$

Proof: Fix x, and let $\mathfrak{M}$ be the set of all y for which the assertion holds.

I)
$$1 \neq x',$$
$$1 \neq x + 1;$$

1 belongs to $\mathfrak{M}$.

II) If y belongs to $\mathfrak{M}$, then

$$y \neq x + y,$$

hence

$$y' \neq (x + y)',$$
$$y' \neq x + y',$$

so that y' belongs to $\mathfrak{M}$.

Therefore the assertion holds for all y.

Theorem 8: *If*

$$y \neq z$$

then

$$x + y \neq x + z.$$

Proof: Consider a fixed y and a fixed z such that

$$y \neq z,$$

and let $\mathfrak{M}$ be the set of all x for which

$$x + y \neq x + z.$$

I)
$$y' \neq z',$$
$$1 + y \neq 1 + z;$$

hence 1 belongs to $\mathfrak{M}$.

II) If x belongs to $\mathfrak{M}$, then

$$x + y \neq x + z,$$

hence

$$(x + y)' \neq (x + z)',$$
$$x' + y \neq x' + z,$$

so that x' belongs to $\mathfrak{M}$.

Therefore the assertion holds always.

Theorem 9: *For given x and y, exactly one of the following must be the case:*

1) $$x = y.$$

2) *There exists a u* (exactly one, by Theorem 8) *such that*

$$x = y + u.$$

3) *There exists a v* (exactly one, by Theorem 8) *such that*

$$y = x + v.$$

Proof: A) By Theorem 7, cases 1) and 2) are incompatible. Similarly, 1) and 3) are incompatible. The incompatibility of 2) and 3) also follows from Theorem 7; for otherwise, we would have

$$x = y + u = (x + v) + u = x + (v + u) = (v + u) + x.$$

Therefore we can have at most one of the cases 1), 2) and 3).

B) Let x be fixed, and let $\mathfrak{M}$ be the set of all y for which one (hence by A), exactly one) of the cases 1), 2) and 3) obtains.

I) For $y = 1$, we have by Theorem 3 that either

$$x = 1 = y \qquad \text{(case 1))}$$

or

$$x = u' = 1 + u = y + u \qquad \text{(case 2))}.$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let y belong to $\mathfrak{M}$. Then

either (case 1) for y)

$$x = y,$$

hence

$$y' = y + 1 = x + 1 \qquad \text{(case 3) for } y');$$

or (case 2) for y)

$$x = y + u,$$

hence if

$$u = 1,$$

then

$$x = y + 1 = y' \qquad \text{(case 1) for } y');$$

but if

$$u \neq 1,$$

then, by Theorem 3,

$$u = w' = 1 + w,$$

$$x = y + (1 + w) = (y + 1) + w = y' + w$$

(case 2) for y');

or (case 3) for y)

$$y = x + v,$$

hence

$$y' = (x + v)' = x + v'$$

(case 3) for y').

In any case, y' belongs to $\mathfrak{M}$.

Therefore we always have one of the cases 1), 2) and 3).

---

§ 3

Ordering

Definition 2: *If*

$$x = y + u$$

then

$$x > y.$$

($>$ to be read "is greater than.")

Definition 3: *If*

$$y = x + v$$

then

$$x < y.$$

($<$ to be read "is less than.")

Theorem 10: *For any given x, y, we have exactly one of the cases*

$$x = y, \quad x > y, \quad x < y.$$

Proof: Theorem 9, Definition 2 and Definition 3.

Theorem 11: *If*

$$x > y$$

then

$$y < x.$$

Proof: Each of these means that

$$x = y + u$$

for some suitable u.

Theorem 12: *If*

$$x < y$$

then

$$y > x.$$

Proof: Each of these means that

$$y = x + v$$

for some suitable v.

Definition 4: $x \geqq y$

means

$$x > y \ \text{ or } \ x = y.$$

($\geqq$ to be read "is greater than or equal to.")

Definition 5: $x \leqq y$

means

$$x < y \quad or \quad x = y.$$

($\leqq$ to be read "is less than or equal to.")

Theorem 13: *If*

$$x \geqq y$$

then

$$y \leqq x.$$

Proof: Theorem 11.

Theorem 14: *If*

$$x \leqq y$$

then

$$y \geqq x.$$

Proof: Theorem 12.

Theorem 15 (Transitivity of Ordering): *If*

$$x < y, \quad y < z,$$

then

$$x < z.$$

Preliminary Remark: Thus if

$$x > y, \quad y > z,$$

then

$$x > z,$$

since

$$z < y, \quad y < x,$$
$$z < x;$$

but in what follows I will not even bother to write down such statements, which are obtained trivially by simply reading the formulas backwards.

Proof: With suitable v, w, we have

$$y = x + v, \quad z = y + w,$$

hence

$$z = (x + v) + w = x + (v + w),$$
$$x < z.$$

Theorem 16: *If*

$$x \leqq y, \quad y < z \quad or \quad x < y, \quad y \leqq z,$$

then

$$x < z.$$

Proof: Obvious if an equality sign holds in the hypothesis: otherwise, Theorem 15 does it.

Theorem 17: *If*

$$x \leqq y, \quad y \leqq z,$$

then

$$x \leqq z.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise, Theorem 16 does it.

A notation such as

$$a < b \leqq c < d$$

is justified on the basis of Theorems 15 and 17. While its immediate meaning is

$$a < b, \quad b \leqq c, \quad c < d,$$

it also implies, according to these theorems, that, say

$$a < c, \quad a < d, \quad b < d.$$

(Similarly in the later chapters.)

Theorem 18: $x + y > x.$

Proof: $x + y = x + y.$

Theorem 19: *If*

$$x > y, \ or \ x = y, \ or \ x < y,$$

then

$$x + z > y + z, \ or \ x + z = y + z, \ or \ x + z < y + z,$$

respectively.

Proof: 1) If

$$x > y$$

then

$$x = y + u,$$
$$x + z = (y + u) + z = (u + y) + z = u + (y + z) = (y + z) + u,$$
$$x + z > y + z.$$

2) If

$$x = y$$

then clearly

$$x + z = y + z.$$

3) If

$$x < y$$

then

$$y > x,$$

hence, by 1),

$$y + z > x + z,$$
$$x + z < y + z.$$

Theorem 20: *If*

$$x + z > y + z, \ or \ x + z = y + z, \ or \ x + z < y + z,$$

then

$$x > y, \text{ or } x = y, \text{ or } x < y, \text{ respectively.}$$

Proof: Follows from Theorem 19, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 21: *If*

$$x > y, \quad z > u,$$

then

$$x + z > y + u.$$

Proof: By Theorem 19, we have

$$x + z > y + z$$

and

$$y + z = z + y > u + y = y + u,$$

hence

$$x + z > y + u.$$

Theorem 22: *If*

$$x \geqq y, \; z > u \text{ or } x > y, \; z \geqq u,$$

then

$$x + z > y + u.$$

Proof: Follows from Theorem 19 if an equality sign holds in the hypothesis, otherwise from Theorem 21.

Theorem 23: *If*

$$x \geqq y, \; z \geqq u,$$

then

$$x + z \geqq y + u.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise Theorem 22 does it.

Theorem 24: $x \geqq 1.$

Proof: Either

$$x = 1$$

or

$$x = u' = u + 1 > 1.$$

Theorem 25: *If*

$$y > x$$

then

$$y \geqq x + 1.$$

Proof:

$$y = x + u,$$

$$u \geqq 1,$$

hence

$$y \geqq x + 1.$$

Theorem 26: *If*

$$y < x + 1$$

then

$$y \leqq x.$$

Proof: Otherwise we would have

$$y > x$$

and therefore, by Theorem 25,

$$y \geqq x + 1.$$

Theorem 27: *In every non-empty set of natural numbers there is a least one* (i.e. one which is less than any other number of the set).

Proof: Let $\mathfrak{N}$ be the given set, and let $\mathfrak{M}$ be the set of all x which are $\leqq$ every number of $\mathfrak{N}$.

By Theorem 24, the set $\mathfrak{M}$ contains the number 1. Not every x belongs to $\mathfrak{M}$; in fact, for each y of $\mathfrak{N}$ the number $y + 1$ does not belong to $\mathfrak{M}$, since

$$y + 1 > y.$$

Therefore there is an m in $\mathfrak{M}$ such that $m + 1$ does not belong to $\mathfrak{M}$; for otherwise, every natural number would have to belong to $\mathfrak{M}$, by Axiom 5.

Of this m I now assert that it is $\leqq$ every n of $\mathfrak{N}$, and that it belongs to $\mathfrak{N}$. The former we already know. The latter is established by an indirect argument, as follows: If m did not belong to $\mathfrak{N}$, then for each n of $\mathfrak{N}$ we would have

$$m < n,$$

hence, by Theorem 25,

$$m + 1 \leqq n;$$

thus $m + 1$ would belong to $\mathfrak{M}$, contradicting the statement above by which m was introduced.

$$X \cdot 1 = X,$$

since

$$\frac{x_1}{x_2} \cdot \frac{1}{1} \sim \frac{x_1 \cdot 1}{x_2 \cdot 1} \sim \frac{x_1}{x_2}.$$

Theorem 114: *If Z is the rational number corresponding to the fraction* $\frac{x}{y}$, *then*

$$yZ = x.$$

Proof:

$$\frac{y}{1}\,\frac{x}{y} \sim \frac{yx}{1 \cdot y} \sim \frac{xy}{1 \cdot y} \sim \frac{x}{1}.$$

Definition 27: *The U of Theorem 110 is called the quotient of X by Y, or the rational number obtained from division of X by Y. It will be denoted by* $\frac{X}{Y}$ (to be read "*X* over *Y*").

Let *X* and *Y* be integers, say $X = x$ and $Y = y$. Then by Theorem 114, the rational number $\frac{x}{y}$ determined by Definitions 26 and 27 stands for the class to which the fraction $\frac{x}{y}$ (in the earlier sense) belongs.

We need not be afraid of confusing the two symbols $\frac{x}{y}$, since fractions as such will from now on no longer occur. $\frac{x}{y}$ will henceforth always denote a rational number. Conversely, every rational number may be expressed in the form $\frac{x}{y}$, by Theorem 114 and Definition 27.

Theorem 115: *Let X and Y be given. Then there exists a z such that*

$$zX > Y.$$

Proof: $\frac{Y}{X}$ is a rational number; by Theorem 89, there exist integers (in our new terminology), say *z* and *v*, such that

$$\frac{z}{v} > \frac{Y}{X}.$$

By Theorem 111, we have

$$v \geqq 1,$$

hence, by Theorem 105,

$$zX = Xz = X\left(\frac{z}{v}v\right) = \left(X\frac{z}{v}\right)v \geqq \left(X\frac{z}{v}\right) \cdot 1 = X\frac{z}{v} > X\,\frac{Y}{X} = Y.$$

CHAPTER III

CUTS

§ 1

Definition

Definition 28: *A set of rational numbers is called a cut if*

1) *it contains a rational number, but does not contain all rational numbers;*

2) *every rational number of the set is smaller than every rational number not belonging to the set;*

3) *it does not contain a greatest rational number* (i.e. a number which is greater than any other number of the set).

We will also use the term "lower class" for such a set, and the term "upper class" for the set of all rational numbers which are not contained in the lower class. The elements of the two sets will then be called "lower numbers" and "upper numbers," respectively.

Small Greek letters will be used throughout to denote cuts, except where otherwise specified.

Definition 29:

$$\xi = \eta$$

(= to be read "is equal to") *if every lower number for* ξ *is a lower number for* η *and every lower number for* η *is a lower number for* ξ.

In other words, if the sets are identical.

Otherwise,

$$\xi \neq \eta$$

($\neq$ to be read "is not equal to").

The following three theorems are trivial:

Theorem 116:

$$\xi = \xi.$$

Theorem 117: *If*

$$\xi = \eta$$

then

$$\eta = \xi.$$

Theorem 118: *If*

$$\xi = \eta,\ \eta = \zeta,$$

then

$$\xi = \zeta.$$