Math 534A Solution to Problem 0.12

As promised at my office hour last week, here's the solution to problem 12 from the Introduction in Marsden & Hoffmann.

Problem: Let $\mathcal{A}$ be a collection of subsets of a set $S$ and $\mathcal{B}$ the collection of complementary sets; that is, $B \in \mathcal{B}$ iff $S \setminus B \in \mathcal{A}$. Prove de Morgan's laws:

a. $S \setminus \bigcup \mathcal{A} = \bigcap \mathcal{B}$.

Proof: We begin by noting that

$$\bigcup \mathcal{A} = \{a \in S \mid \exists A \in \mathcal{A}, a \in A\}. \tag{1}$$

Then by the rules of negation,

$$S \setminus (\bigcup \mathcal{A}) = \{a \in S \mid \forall A \in \mathcal{A}, a \notin A\} \tag{2}$$

But

$$\bigcap \mathcal{B} = \{a \in S \mid \forall B \in \mathcal{B}, a \in B\} \tag{3}$$

and by using the definition of $\mathcal{B}$ this becomes

$$\bigcap \mathcal{B} = \{a \in S \mid \forall (S \setminus B) \in \mathcal{A}, a \in B\} \tag{4}$$

or

$$\bigcap \mathcal{B} = \{a \in S \mid \forall (S \setminus B) \in \mathcal{A}, a \notin (S \setminus B)\}. \tag{5}$$

Renaming $S \setminus B$, $A$ gives the result. This is justified since in the condition $S \setminus B$ runs over all subsets of $S$.

b. $S \setminus \bigcap \mathcal{A} = \bigcup \mathcal{B}$.

Proof: Noting that $S \setminus (S \setminus X) = X$ for all subsets $X$ of $S$, this follows by taking complements in part a, and switching the roles of the two collections.