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# Chapter 6 Sensitivity Analysis and Duality

to accompany Introduction to Mathematical Programming: Operations Research, Volume 1 4th edition, by Wayne L. Winston and Munirpallam Venkataramanan

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## 6.5 – Finding the Dual of an LP

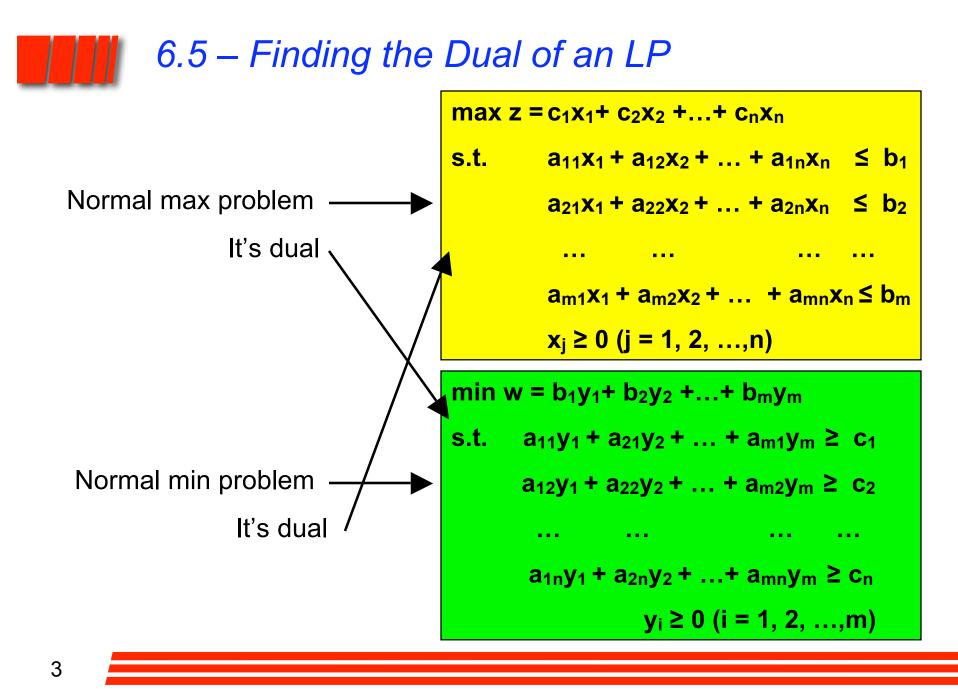
Associated with any LP is another LP called the **dual**. Knowledge of the dual provides interesting economic and sensitivity analysis insights.

When taking the dual of any LP, the given LP is referred to as the **primal**. If the primal is a max problem, the dual will be a min problem and visa versa.

Define the variables for a max problem to be z,  $x_1$ ,  $x_2$ , ...,  $x_n$  and the variables for a min problem to be w,  $y_1$ ,  $y_2$ , ...,  $y_n$ .

Finding the dual to a max problem in which all the variables are required to be nonnegative and all the constraints are  $\leq$  constraints (called normal max problem) is shown on the next slide.

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#### Interpreting the Dual of the Dakota (Max) Problem

The primal is: max $z = 60x_1 + 30x_2 + 20x_3$							
	s.t. 8x <sub>1</sub> + 6x <sub>2</sub> + x <sub>3</sub> ≤ 48	(Lumber constraint)					
	4x <sub>1</sub> + 2x <sub>2</sub> + 1.5x <sub>3</sub> ≤ 20	(Finishing constraint)					
	2x <sub>1</sub> + 1.5x <sub>2</sub> + 0.5x <sub>3</sub> ≤ 8	(Carpentry constraint)					
	$x_1, x_2, x_3 \ge 0$						
The dual is: min w = $48y_1 + 20y_2 + 8y_3$							
	s.t. $8y_1 + 4y_2 + 2y_3 \ge 60$	(Desk constraint)					
	$6y_1 + 2y_2 + 1.5y_3 \ge 30$	(Table constraint)					
	y <sub>1</sub> + 1.5y <sub>2</sub> + 0.5y <sub>3</sub> ≥ 20	(Chair constraint)					

 $y_1 + 1.5y_2 + 0.5y_3 \ge 20$ 

 $y_1, y_2, y_3 \ge 0$ 

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The dual is:	min w = 48 $y_1$ + 20 $y_2$ +8 $y_3$				
	s.t. $8y_1 + 4y_2 + 2y_3 \ge 60$	(Desk constraint)			
	6y <sub>1</sub> + 2y <sub>2</sub> + 1.5y <sub>3</sub> ≥ 30	(Table constraint)			
	y <sub>1</sub> + 1.5y <sub>2</sub> + 0.5y <sub>3</sub> ≥ 20	(Chair constraint)			
	y <sub>1</sub> , y <sub>2</sub> , y <sub>3</sub> ≥ 0				

Relevant information about the Dakota problem dual is shown below.

Resource	Desk	Table	Chair	Availability
Lumber	8 board ft	6 board ft	1 board ft	48 boards ft
Finishing	4 hours	2 hours	1.5 hours	20 hours
Carpentry	2 hours	1.5 hours	0.5 hours	8 hours
Selling Price	\$60	\$30	\$20	

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The first dual constraint is associated with desks, the second with tables, and the third with chairs. Decision variable  $y_1$  is associated with lumber,  $y_2$  with finishing hours, and  $y_3$  with carpentry hours.

Suppose an entrepreneur wants to purchase all of Dakota's resources. The entrepreneur must determine the price he or she is willing to pay for a unit of each of Dakota's resources.

To determine these prices we define:

 $y_1$  = price paid for 1 boards ft of lumber

 $y_2$  = price paid for 1 finishing hour

 $y_3$  = price paid for 1 carpentry hour

The resource prices  $y_1$ ,  $y_2$ , and  $y_3$  should be determined by solving the Dakota dual.



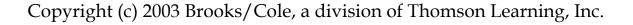
The total price that should be paid for these resources is  $48 y_1 + 20y_2 + 8y_3$ . Since the cost of purchasing the resources is to minimized:

min w = 
$$48y_1 + 20y_2 + 8y_3$$

is the objective function for the Dakota dual.

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In setting resource prices, the prices must be high enough to induce Dakota to sell. For example, the entrepreneur must offer Dakota at least \$60 for a combination of resources that includes 8 board feet of lumber, 4 finishing hours, and 2 carpentry hours because Dakota could, if it wished, use the resources to produce a desk that could be sold for \$60. Since the entrepreneur is offering  $8y_1 + 4y_2 + 2y_3$  for the resources used to produce a desk, he or she must chose  $y_1$ ,  $y_2$ , and  $y_3$  to satisfy:  $8y_1 + 4y_2 + 2y_3 \ge 60$ 



Similar reasoning shows that at least \$30 must be paid for the resources used to produce a table. Thus  $y_1$ ,  $y_2$ , and  $y_3$  must satisfy:

 $6y_1 + 2y_2 + 1.5y_3 \ge 30$ 

Likewise, at least \$20 must be paid for the combination of resources used to produce one chair. Thus  $y_1$ ,  $y_2$ , and  $y_3$  must satisfy:

 $y_1 + 1.5y_2 + 0.5y_3 \ge 20$ 

The solution to the Dakota dual yields prices for lumber, finishing hours, and carpentry hours.

In summary, when the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as **resource shadow prices**.