Chapter 6
Sensitivity Analysis and Duality

to accompany

Introduction to Mathematical Programming: Operations Research, Volume 1

4th edition, by Wayne L. Winston and Munirpallam Venkataramanan

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6.5 – Finding the Dual of an LP

Associated with any LP is another LP called the **dual**. Knowledge of the dual provides interesting economic and sensitivity analysis insights.

When taking the dual of any LP, the given LP is referred to as the **primal**. If the primal is a max problem, the dual will be a min problem and visa versa.

Define the variables for a max problem to be $z, x_1, x_2, \ldots, x_n$ and the variables for a min problem to be $w, y_1, y_2, \ldots, y_n$.

Finding the dual to a max problem in which all the variables are required to be nonnegative and all the constraints are ≤ constraints (called normal max problem) is shown on the next slide.

6.5 – Finding the Dual of an LP

Normal max problem

It's dual

$$\max z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\begin{aligned}
\text{s.t.}\quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
& \ldots \quad \ldots \quad \ldots \quad \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
& x_j \ge 0 \ (j = 1, 2, \ldots, n)
\end{aligned}$$

Normal min problem

It's dual

$$\min w = b_1y_1 + b_2y_2 + \ldots + b_my_m$$

$$\begin{aligned}
\text{s.t.}\quad & a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1 \\
& a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2 \\
& \ldots \quad \ldots \quad \ldots \quad \ldots \\
& a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n \\
& y_i \ge 0 \ (i = 1, 2, \ldots, m)
\end{aligned}$$

6.5 – Economic Interpretation of the Dual Problem

Interpreting the Dual of the Dakota (Max) Problem

The primal is: $\max z = 60x_1 + 30x_2 + 20x_3$

$$
\begin{aligned}
\text{s.t.}\quad 8x_1 + 6x_2 + x_3 &\le 48 && \text{(Lumber constraint)}\\
4x_1 + 2x_2 + 1.5x_3 &\le 20 && \text{(Finishing constraint)}\\
2x_1 + 1.5x_2 + 0.5x_3 &\le 8 && \text{(Carpentry constraint)}\\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

The dual is: $\min w = 48y_1 + 20y_2 + 8y_3$

$$
\begin{aligned}
\text{s.t.}\quad 8y_1 + 4y_2 + 2y_3 &\ge 60 && \text{(Desk constraint)}\\
6y_1 + 2y_2 + 1.5y_3 &\ge 30 && \text{(Table constraint)}\\
y_1 + 1.5y_2 + 0.5y_3 &\ge 20 && \text{(Chair constraint)}\\
y_1, y_2, y_3 &\ge 0
\end{aligned}
$$

6.5 – Economic Interpretation of the Dual Problem

The dual is: min $w = 48y_1 + 20y_2 + 8y_3$

s.t. $8y_1 + 4y_2 + 2y_3 \geq 60$ (Desk constraint)

$6y_1 + 2y_2 + 1.5y_3 \geq 30$ (Table constraint)

$y_1 + 1.5y_2 + 0.5y_3 \geq 20$ (Chair constraint)

$y_1, y_2, y_3 \geq 0$

Relevant information about the Dakota problem dual is shown below.

Resource	Desk	Table	Chair	Availability
Lumber	8 board ft	6 board ft	1 board ft	48 boards ft
Finishing	4 hours	2 hours	1.5 hours	20 hours
Carpentry	2 hours	1.5 hours	0.5 hours	8 hours
Selling Price	$60	$30	$20	

6.5 – Economic Interpretation of the Dual Problem

The first dual constraint is associated with desks, the second with tables, and the third with chairs. Decision variable y_1 is associated with lumber, y_2 with finishing hours, and y_3 with carpentry hours.

Suppose an entrepreneur wants to purchase all of Dakota's resources. The entrepreneur must determine the price he or she is willing to pay for a unit of each of Dakota's resources.

To determine these prices we define:

y_1 = price paid for 1 boards ft of lumber

y_2 = price paid for 1 finishing hour

y_3 = price paid for 1 carpentry hour

The resource prices y_1, y_2, and y_3 should be determined by solving the Dakota dual.

6.5 – Economic Interpretation of the Dual Problem

The total price that should be paid for these resources is $48y_1 + 20y_2 + 8y_3$. Since the cost of purchasing the resources is to minimized:

$$\min w = 48y_1 + 20y_2 + 8y_3$$

is the objective function for the Dakota dual.

In setting resource prices, the prices must be high enough to induce Dakota to sell. For example, the entrepreneur must offer Dakota at least \$60 for a combination of resources that includes 8 board feet of lumber, 4 finishing hours, and 2 carpentry hours because Dakota could, if it wished, use the resources to produce a desk that could be sold for \$60. Since the entrepreneur is offering $8y_1 + 4y_2 + 2y_3$ for the resources used to produce a desk, he or she must chose y_1, y_2, and y_3 to satisfy: $8y_1 + 4y_2 + 2y_3 \geq 60$

6.5 – Economic Interpretation of the Dual Problem

Similar reasoning shows that at least $30 must be paid for the resources used to produce a table. Thus y_1, y_2, and y_3 must satisfy:

$$6y_1 + 2y_2 + 1.5y_3 \geq 30$$

Likewise, at least $20 must be paid for the combination of resources used to produce one chair. Thus y_1, y_2, and y_3 must satisfy:

$$y_1 + 1.5y_2 + 0.5y_3 \geq 20$$

The solution to the Dakota dual yields prices for lumber, finishing hours, and carpentry hours.

In summary, when the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as **resource shadow prices**.