

PREFACE

The material in this manual ranges from comments on the text and problems to complete solutions of problems. It can be used by teachers to guide class discussion of problems, to provide hints, or to provide a starting point for further discussion. It is practically a necessity for students who will not be discussing problems in class.

I hope this material will not be used to stifle creativity. Consequently I have avoided the more open ended questions containing words like "criticized," "defend," and "suggest." Remember that you may come up with a model that is better than one given here. Except in mathematical calculations, it's a matter of better and worse, not right and wrong.

SOLUTIONS MANUAL
TO
AN INTRODUCTION
TO
MATHEMATICAL MODELING

Edward A. Bender
University of California, San Diego

A Wiley-Interscience Publication
JOHN WILEY & SONS
New York Chichester Brisbane Toronto

Copyright © 1978 by John Wiley & Sons, Inc.

All rights reserved. Published simultaneously in Canada.

No part of this book may be reproduced by any means,
nor transmitted, nor translated into a machine language
without the written permission of the publisher.

ISBN 0-471 03407-x

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

CONTENTS

PREFACE.	vii
1. WHAT IS MODELING.	1
2. ARGUMENTS FROM SCALE.	4
3. GRAPHICAL METHODS	7
4. BASIC OPTIMIZATION.	11
5. BASIC PROBABILITY	17
6. POTPOURRI	20
7. (NO MODELS IN THIS CHAPTER)	
8. QUANTITATIVE DIFFERENTIAL EQUATIONS	22
9. LOCAL STABILITY THEORY.	27
10. STOCHASTIC MODELS	31

CHAPTER 1

WHAT IS MODELING

As mentioned in Section 1.2, it is hard to say whether theory or examples should come first. I decided on the theory. In lecturing you may wish to interchange Sections 1.2 and 1.3 with Sections 1.4 and 1.5. Also, you may wish to go deeper into one of these two problems through lecture or class discussion. Because of my own interests, I talked about the population growth problem quite a bit. The simplest form for $r(N)$ is a straight line $(1 - N/K)r$. Biologists call r the intrinsic growth rate, K the carrying capacity, and $N' = (1 - N/K)rN$ the logistic equation. Random effects can be introduced in various ways such as (i) decrease N instantaneously (as in a plague) and (ii) vary K (the weather or something else causes the number of organisms the environment can support to fluctuate). At this point students may get carried away by complexity and this provides a good example to emphasize the fact that you may never be able to collect the data a complicated model requires. If you are interested in simple simulation by hand or computer, the difference equation analog of $N' = Nr(N)$ can be used.

1.5. ANOTHER EXAMPLE

PROBLEMS

Problems 1, 2, 5, and 6 can easily lead to projects. Beware of huge projects on #5.

- (a) Any scheme that is considered must be capable of simple implementation and must not lead to confusion among the passengers. For example, suppose two elevators are available and the percentage of traffic to each floor from the ground in the morning rush is

7	30%
6	10%
5	10%
4	30%
3	10%
2	10%

Using one elevator for 2, 3, 4 and another for 5, 6, 7 gives a 50-50 split; but 60% of the riders wait through two stops. The division 2, 3, 7 and 4, 5, 6 is better but may confuse riders.

- (b) There are two problems here: What should be used as a measure? How can it be calculated for the model? Computer simulation maybe useful.

2. The three basic interactions between two species are predator-prey, competition, and symbiosis. See Section 9.2 for a discussion of some ideas. Difference equation models and computer simulation could be introduced here. The predator-prey situation (which includes herbivore-plant and parasite-host) is the most interesting. In constructing a model with actual numbers it is useful to know that predator efficiency is about 10%; i.e., about 10% of the consumed prey weight is used to produce useful energy and protein in the predator.

3. Problem 1 tends to balance modeling and data-collection questions. Discussion of Problem 2 emphasizes modeling considerations. In this problem the emphasis will probably be on data collection since it is fairly obvious that

$$(\text{available energy}) \times (\text{energy used per mile})$$

is a fair approximation to total number of miles. (Approximation only — the second factor depends on body weight, wind conditions, etc., and so is not a constant.)

6. Two plausible criteria for the borderline between legible and illegible are (i) the solid angle subtended by the character and (ii) the linear angle subtended by the character in the horizontal direction. The equations for the borderline can be worked out in each case. The latter gives a circle.
7. Factor analysis, regression, and time series analysis are three techniques commonly used in this way. The results can be useful. We assume that there is some underlying process that is related to a variety of variables. Hence any given variable is related to the others. (No cause and effect relationship is implied by this.) We attempt to estimate crudely what this relationship looks like. We may detect false relationships in this way simply because of the data sample.
8. The text does not go into statistical methods for measuring goodness of fit. This problem illustrates the usefulness of such knowledge. This and later problems may encourage students to take a course in applied statistics.

Even with statistical tools, there is still a problem since the table contains no mention of dN/dt . One approach is to solve the differential equation another is to approximate dN/dt by something like $N(t + \Delta t) - N(t)$. We can avoid a factor of N by noting that $N'/N = (\log N)'$ thus:

$$\log N(t + \Delta t) - \log N(t) \approx \Delta t r(1 + N(\tau)/K)$$

when $t \leq \tau \leq t + \Delta t$. A good choice for τ is $t + \Delta t/2$. To do this we must set Δt equal to twice the time interval in the table. Alternatively, we can approximate $N(\tau)$ by averaging $N(t)$ and $N(t + \Delta t)$. If N grew exponentially, the proper average would be the geometric mean rather than the arithmetic. Hence we take

$$N(\tau) = (N(t) N(t + \Delta t))^{\frac{1}{2}}.$$

CHAPTER 2

ARGUMENTS FROM SCALE

2.1. EFFECTS OF SIZE

Cost of Packaging

The cost of packaging model seems to me a good introductory model because it is easy, involves considerable approximation, and has data collection difficulties. This provides something more concrete and mathematical after the rather vague discussion in Sections 1.4 and 1.5. Among the problems not considered are rounding off prices and unreasonable pricing.

Speed of Racing Shells

In his discussion, McMahon argues that sustained power output is proportional to surface area. Then (5) becomes

$$t_L/t_H = (1.18)^{2/9} (S_L/S_H) = 1.04(S_L/S_H).$$

Size Effects in Animals

Biologists have tried to explain structure of animals in mechanical terms. See K. Schmidt-Nielsen (1972). I would be interested in any studies done on h , A_n , and V_m for jumping mammals.

PROBLEMS

4. A major assumption is that heart volume is proportional to body volume. Then the model gives

$$\text{flow} \propto m^{2/3}$$

$$\text{flow} \propto (\text{pulse}) \times m$$

and so

$$\text{pulse} \propto m^{-1/3}.$$

A better fit to the data is given by $\text{pulse} \propto m^{-1/4}$. This would be obtained if heart volume were proportional to $m^{11/12}$.

5. A variety of conclusions are possible. From (11), $m \propto l^4$, so the Lilliputian reasoning was wrong. From Problem 4 we can try to say something about energy usage E . Either $E \propto m^{2/3}$ or $E \propto m^{3/4}$ can be justified. Hence $E \propto l^{8/3}$ or $E \propto l^3$. The latter is the Lilliputian answer!
6. Fechner assumes that jnd's are additive. It may not be true that a difference of, say, 10 jnd's seems the same at low and high intensities. Steven's law is a power law, $S = aF^c$, which is close to logarithmic for small c .
7. This explains why small animals survive falls from heights better than large animals.

2.2. DIMENSIONAL ANALYSIS

PROBLEMS

1. (b) If the constant of proportionality is κ^* , then $\kappa l/m$ can be replaced by $\kappa^* l/m^2 g$ and so

$$\tau = \sqrt{l/g} F^*(\theta, \kappa^* l/m^2 g)$$

- (c) Adjust l and m so that l/m is constant or l/m^2 is constant. Keep θ fixed. Plot τ versus l on log-log paper.
2. (d) I don't know the answer; however, σ may depend on the diameter of the string and not just the material. See (e).
3. (a) Why should the density of the meat not enter? Heat has to flow a certain distance so the size of the meat is important. The material is important only as far as it affects heat flow, and κ allows for this.
- (c) time/volume $\propto (\text{volume})^{-1/3}$.

$$25/15 = 1.67$$

$$(30/7)^{1/3} = 1.62.$$

The rule seems fairly accurate.

4. (e) If we use the scale modeling idea in the text, then λ

is scaled down by r and g up by r , hence v is unchanged. Time is λ/v and so it is scaled down by r . You might look for methods that avoid a centrifuge. Then g would be constant and time would be scaled down by $r^{1/2}$.

CHAPTER 3

GRAPHICAL METHODS

3.2. COMPARATIVE STATICS

The Nuclear Missile Arms Race

One can argue that many of the curves are piecewise linear: Consider $x=f(y)$. When $y \leq x$, the expected number of survivors is $(x-y) + py$ where p is the probability that a missile will survive attack by a single warhead. In general, for $y = mx + z$, $0 \leq z \leq x$, we expect $(x-z)p^m + zp^{m+1}$ survivors. (See Problem 5.1.5 for further discussion.)

Biogeography: Diversity of Species on Islands

The ideas in this model have been explored quantitatively as well as qualitatively. For example J. M. Diamond (1973. "Distributional Ecology of New Guinea Birds." Science 179, 759-769) fits curves by regression. The curve $S = 12.3A^{0.22}$ where S is the number of bird species and A is area of the island in square kilometers provides a good fit for islands near New Guinea. A correction factor can be introduced for distance from the mainland: $e^{-D/2600}$. Since the largest D is 9200 kilometers, this factor ranges from 0.03 to 1. Finally, he introduces a factor due to habitat diversity resulting from elevation (L in meters). The final formula is

$$S = 1.23A^{0.22} e^{-D/2600} (1 + 8.9 \times 10^{-5}L).$$

The form is, of course, largely ad hoc; however, the fit to the data is fairly good.

PROBLEMS

4. There will be a point at which country 1 has first strike capability but not deterrent capability; that is, it can inflict unacceptable damage if and only if it fires its missiles first. Country 1 may decide to keep country 2 from reaching this level by engaging in "preventive war." If country 2 reaches this level, both countries may be "trigger happy" in tense situations.

If the countries estimate the curve $x=f(y)$, differently, this can cause added complications by introducing a zone where country 2 feels country 1 is capable of deterrence but country 1 feels it has only first strike capability.

3.3. STABILITY QUESTIONS

Cobweb Models in Economics

The data in Table I is worth examining. A demand curve could be gotten by plotting "bushels in year n " versus "deflated price in year n ." A supply curve could be gotten by plotting "acres in year n " times "expected yield per acre" versus "deflated price in year $n-1$." The expected yield per acre is rather hard to determine. Should it be the average yield over several years? the present yield? the previous yield?

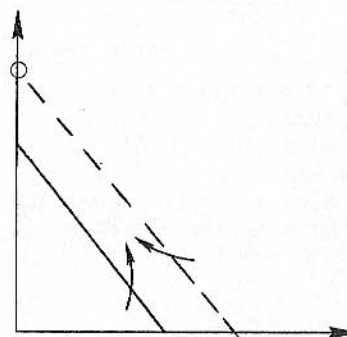
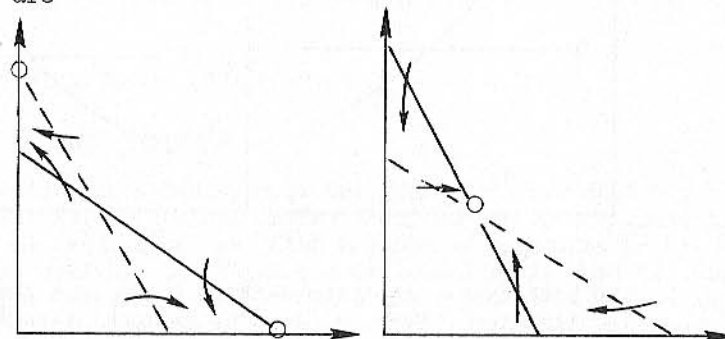
Small-Group Dynamics

The group dynamics model is open to criticisms which might lead to other functional relationships.

PROBLEMS

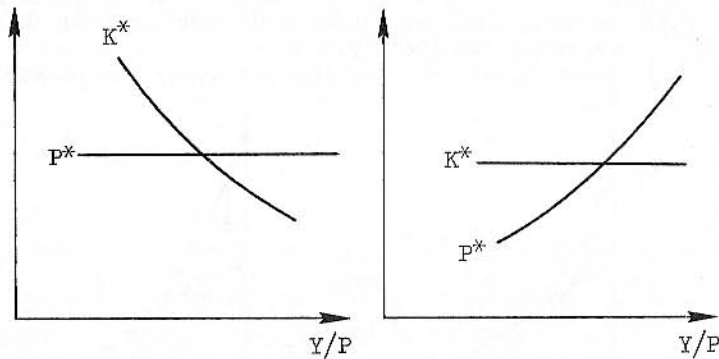
- One can draw cobwebs with n year lags instead of 1 year lags (n equals the number of years required to bring a hog to marketable size). This assumes that the price of feed corn is constant. If the price of feed corn varies, there will be complications. The price may be unrelated to hog production (e.g. due to weather), or it may be closely related to hog production if hog farmers are the principle feed corn buyers. In the latter case, the decision to raise hogs depends on the market price of hogs and the market price of corn. The decision to raise corn depends on the market price of corn which depends on the previous crop and the number of hogs being raised. The net result is that the low corn prices favor hog raising which in turn drives corn prices up. Higher corn prices and a greater supply of hogs cause hog raising to decrease. Eventually hog production and corn prices fall. By studying this carefully, you can get an idea how much corn and hog prices should be offset for a good correlation.
- This can be treated as a cobweb model with a 3 to 5 year delay. The "money" axis should probably not be "starting salary," but perhaps "expected starting salary" (i.e., starting salary times probability of employment). The curve is then practically a hyperbola.

- An alternate approach is via net growth rates: If $x' = x n(x,y)$, then $\partial n / \partial x < 0$. If $x' = g(x,y)$, this is equivalent to $\partial g / \partial x < 0$ when $x' = 0$; however, it is easier to justify.
 - Assuming at most one intersection, the possible graphs are



Solid line: $x' = 0$ curve
 Dashed line: $y' = 0$ curve
 Circle: stable point

5. (c) The two "traps" look like



In both cases, the intersection point is a stable equilibrium. Here Y denotes national income.

CHAPTER 4

BASIC OPTIMIZATION

4.1. OPTIMIZATION BY DIFFERENTIATION

Geometry of Blood Vessels

The assumption of a function of the form Kr^a for cost per unit length is, of course, unjustified, but it does include the cases $a=1$ and $a=2$ and also a range of functions in between. In reality, the thickness of vessel walls does not vary in a simple fashion with r . Since the capillary estimate is fairly accurate, it would be interesting to know how accurate the branching angle prediction is.

Fighting Forest Fires

A model of this sort might prove quite useful. Clearly further study is needed and the objections concerning H/E in the last paragraph of the model need to be overcome. Also the use of equipment such as bulldozers and airplanes should be taken into account. Implementation may well be a problem: One of my students attempted to gather data and found that there was opposition to modeling at the district level.

PROBLEMS

2. (b) A simple interpretation is: If when standing still your back gets at least as wet as the rest of you ($\varphi \leq w$), then run so as to keep your back and front dry ($v=w$); otherwise run as fast as possible.
3. (a) This is the equilibrium situation we are talking about here. For $N > N_f$, it is profitable to fish so that N decreases. For $N < N_f$, fishing is unprofitable, so N will increase.
 - (d) If $N^* < N_f$, no fishing will be done since it is impossible to break even. Hence $N^* > N_f$ can be assumed. To make a profit we must have $p > c(N_p)$ and so, since c is decreasing and $p = c(N_f)$, $N_p > N_f$. We cannot compare N_f and N_m without data, but it is often

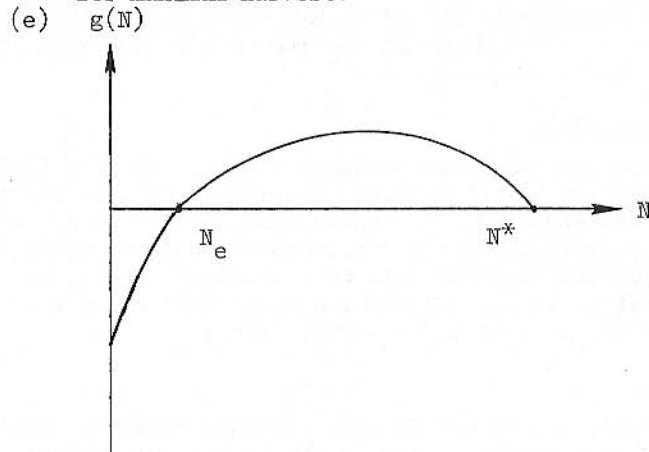
the case that $N_f < N_m$. (This leads to overfishing.)
 What about N_p versus N_m ? By (a), $c' < 0$. Since $g(N_p) > 0$, it follows from

$$p' = (p - c)g' - c'g$$

that

$$(p - c(N_p))g'(N_p) < 0.$$

Since fishing is assumed profitable, $p > c(N_p)$ and so $g'(N_p) < 0$. Thus $N_p > N_m$. In other words, profit is maximized at a higher fish population than is needed for maximum harvest.



If the population falls below N_e , extinction will follow. This will happen if $N_f < N_e$ and fishing is not controlled.

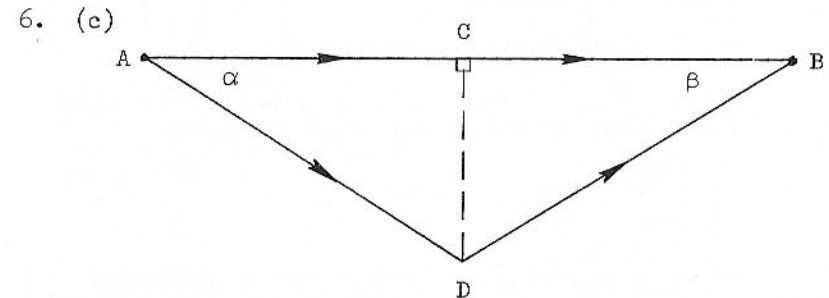
4. (d) Differentiating the expression in (b) with respect to M_j and setting the result equal to zero we have

$$-\frac{b}{M_{j-1} - F_{j-1}} + \frac{1}{M_j} + \frac{a-1}{M_j - F_j} = 0$$

where $\partial F_{j-1}/\partial M_j$ and $\partial F_j/\partial M_j$ were found using (c).
 Multiplying by M_j and rearranging:

$$\begin{aligned} \frac{(a-1)M_j}{M_j - F_j} &= \frac{bM_j}{M_{j-1} - F_{j-1}} - 1 \\ &= \frac{(a-1)M_{j-1}}{M_{j-1} - F_{j-1}} \text{ by (c).} \end{aligned}$$

This says Δv is the same for each stage.



Direct swimming from A to B is along \overline{ACB} . A single burst swim is along \overline{ADB} . Let $CD = h$. Then

$$\overline{AC} = h \cot \alpha$$

$$\overline{CB} = h \cot \beta$$

$$\overline{DB} = h \csc \beta.$$

The energy used along \overline{ACB} is

$$kD(h \cot \alpha + h \cot \beta)$$

and along \overline{ADB} ,

$$(kD + W \sin \beta) h \csc \beta.$$

Combine these with $W \sin \alpha = D$.

- (d) Remember $k = 3$.

- (e) The following table of the ratio in (c) with $\alpha = 11.3^\circ$ and $k = 3$ shows that almost any sort of burst swimming is useful

β	10°	30°	50°	60°	70°	90°
ratio	0.70	0.55	0.51	0.51	0.52	0.54

7. (c) If the net revenue per unit time provided by the entire market (advertising excluded) is r , then the profit of Y per unit time is

$$P = rf(y/(y+z)) - y.$$

Since Y can adjust y , we set $\partial P/\partial y = 0$ to obtain

$$f'(y/(y+z)) = rz/(y+z)^2.$$

From Z 's viewpoint:

$$f'(z/(y+z)) = rz/(y+z)^2,$$

and so $y=z$ by (b). This is necessary for equilibrium but not sufficient since at $y=z$

$$\frac{\partial P}{\partial y} = \frac{rf'(\frac{1}{2})}{4y} - 1,$$

which is generally not zero. Setting this equal to zero we obtain the equilibrium level of advertising

$$y = rf'(\frac{1}{2})/4.$$

We can expect r to be proportional to the size of the market. The effectiveness of advertising is measured by f' . Using this equation and $f'(\frac{1}{2}) = \frac{1}{2}$, we have

$$P = r(2 - f'(\frac{1}{2}))/4.$$

If advertising is too effective ($f'(\frac{1}{2}) > 2$) the firms will go broke due to advertising. Why doesn't this happen normally?

8. Let

T_y = trade in value of y year old vehicle

C = cost of new vehicle

M_y = annual maintenance cost of y year old vehicle.

Keeping a vehicle costs M_y . Buying a new vehicle costs

$C + M_1 - T_y$, so we keep the vehicle if $C + M_1 > T_y + M_y$.

But this only considers the present year. Over the life of a vehicle we spend

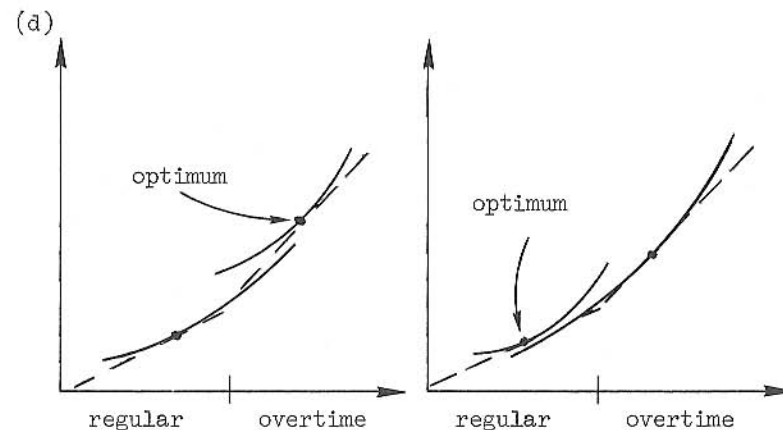
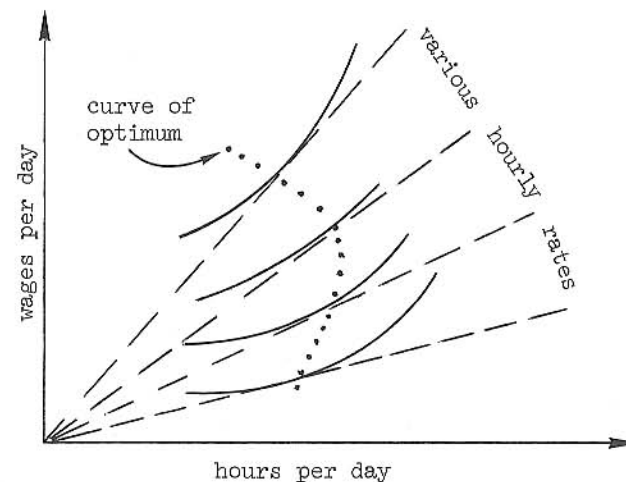
$$\sum_{y=1}^Y M_y + C - T_Y.$$

We could divide by Y and minimize. Discounting can be introduced, too.

4.2. GRAPHICAL METHODS

PROBLEMS

3. (a,b)



5. The constraints provide regions within the temperature pressure plane that the process must lie in. Graph curves of constant yield to find the optimal point in the

intersection of the regions.

6. This can be phrased as follows: Given a triangle and a point, there exists another point which is closer than the given point to at least two vertices of the triangle. This can be generalized to show that with v voters and any number of issues, the second politician can win at least $v/2$ votes. This breaks down if there are three politicians. Can this explain why politicians are more outspoken when there are more than two major parties?

CHAPTER 5

BASIC PROBABILITY

5.1. ANALYTICAL MODELS

Sex Preference and Sex Ratio

That the approximation in (1) is accurate for large populations follows from Chebyshev's inequality: Consider N_i 's fixed and set $X = \sum M_i$ and $c = (\sum N_i)^{2/3}$ in

$$\Pr\{|X - E(X)| > c\} \leq \frac{\sigma^2(X)}{c^2}.$$

Using $\sigma^2 \leq \sum N_i$ and dividing by $\sum N_i$, it follows that $\sum M_i / \sum N_i$ is nearly always close to $\sum p_i N_i / \sum N_i$ when $\sum N_i$ is large. Now let the N_i 's vary and use a similar argument to show that $\sum N_i / E(\sum N_i)$ is close to 1. Actually what we have been doing is proving results related to the law of large numbers (not discussed here).

PROBLEMS

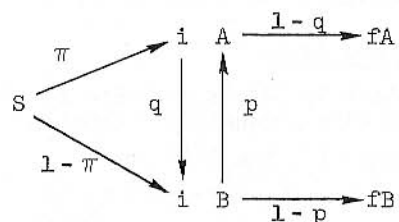
2. (c) Suppose we look for an A decision. Then the last $K+1$ symbols are one B followed by K choices of A. The previous choices can be almost any string without a K long repeat. This argument does not work well for small K because "almost any string without a K long repeat" must exclude strings ending in $K-1$ copies of B. A similar argument applies to a B decision, so P_A/P_B is nearly $qp^K/pq^K = (p/q)^{K-1}$. It can be shown that

$$P_A/P_B = (p/q)^{K-1}(1-q^K)/(1-p^K).$$

- (d) When K varies, longer decision times tend to be associated with larger K . Use the result in (c) on P_A/P_B . For p variable, longer decision times tend to be associated with smaller p . Use (c) to conclude

that longer decision times mean less accuracy in this case. The data cited in the problem fit with variable p but not with variable K (unless p is variable, too).

3. This can be treated as a Markov chain. I will not do so. Let the transition probabilities be as shown here.



The probability of ending in fA is

$$(1) \quad \pi \sum_{n=0}^{\infty} (qp)^n (1-q) + (1-\pi)p \sum_{n=0}^{\infty} (qp)^n (1-q) \\ = (\pi + p - \pi p)(1-q)/(1-pq).$$

Similarly the probability of ending in fB is

$$(2) \quad (1 - \pi + \pi q)(1-p)/(1-pq).$$

In Kintsch's situation, $p=q$ and (1) and (2) become

$$(\pi + p - \pi p)/(1+p)$$

$$(1 - \pi + \pi p)/(1+p)$$

respectively. Their ratio is

$$(\pi + p - \pi p)/(1 - \pi + \pi p).$$

This varies from $\pi/(1-\pi)$ when $p=0$ (no indecision) to 1 as $p \rightarrow 1$ (great indecision). Longer times are associated with less accuracy. You may wish to compute $E(L_A)$ and $E(L_B)$. The former is

$$E(L_A) = \frac{2(\pi + p - \pi p) + (1-\pi)p(1-pq)}{(\pi + p - \pi p)(1-pq)}.$$

4. (b) Actually the formula here is not quite accurate because, as in the sex preference model, $E(u/v) \neq E(u)/E(v)$.

7. (b) A bit of care is needed. Equilibrium implies that net flow into groups of size i is zero, but this includes flow between sizes i and $i-1$ as well as $i+1$ and i . Hence we have only that these latter two net flows are equal. Since this holds for all i , all such flows have the same sign. Positive implies decreasing average group size and negative implies increasing average group size. Both of these are impossible.
- (e) Worse since less data is used in the estimate.

5.2. MONTE CARLO SIMULATION

PROBLEMS

1. The spherical particle can be allowed to move downward until it either touches bottom, a side and two spheres, or three spheres. The programming is more complicated than that for the model in the text.
4. It would be a good idea to look at Hammersley's article. Briefly the idea is as follows. After deriving (b) determine scaling by using data on orbit perturbations of known comets. Hammersley obtains a time unit of 50,000 years. We can use observed orbit data to estimate how many comets are likely to be lost in their present circuit. Hammersley estimates between $1/4$ and $1/2$ are lost annually. Using (b) we can compute the average lifetime of a comet and combine this with the loss rate to get the number present.
5. In this problem it is easy to get bogged down dealing with implementation instead of spending adequate time on model formulation. For example, should erosion be allowed for? Is the general overall slope in (c) reasonable for large drainage basins? Mountainous areas?

CHAPTER 6

POTPOURRI

PROBLEMS

1. Let $n_1(t)$ be the number of revolutions of the take-up reel after t feet of tape have been played and let $n_2(t)$ be read from the runoff counter. We wish to relate n_1 and n_2 . Let's focus on the take-up reel. When viewed with the tape edgewise, the area of the tape is proportional to its length and the thickness of the ring is proportional to the number of revolutions made by the reel. Solving the two resulting equations for t in terms of n_1 we obtain

$$Cn_1^2 + Dn_1 = t$$

where C and D depend on physical characteristics of the reel and tape. If N is the number of revolutions required to empty a reel, it follows that

$$C(N - n_2)^2 + D(N - n_2) = T - t.$$

Combining these two equations we obtain the circle

$$(n_1 - c)^2 + (N - n_2 - c)^2 = r^2.$$

If M is the number of revolutions needed to half-fill a reel, then using the two points $(n_1, n_2) = (0, 0)$ and $(M, N - M)$ we obtain

$$c = \frac{2M^2 - N^2}{4M - 2N} \quad \text{and} \quad r = \sqrt{2} \frac{2M^2 - 2MN + N^2}{4M - 2N}.$$

2. Levary deals with a somewhat different problem: He assumes that there is only one of the slow moving items and that we can expect to wait t years to sell it. He obtains the condition

$$(1) \quad L^* + L^*pNt \geq L,$$

the left side being our income after t years if we sell the item now at L^* and the right being our income if we do not sell it. One could argue that the left side should be compounded thus

$$(2) \quad L^*(1+p)^{Nt} \geq L.$$

Our problem is more complicated since we have more than one of the slow moving items: We cannot multiply both sides of Levary's result by this number since their sale will be spread over a period of time. If there are s slow moving items, then the average length of time on the shelf is $t = s/2S$ and (1) becomes

$$s(L^* + L^*pNs/S) \geq s(L + LpNs/2S)$$

and (2) becomes

$$sL^*(1+p)^{Ns/S} \geq L \int_0^{s/S} (1+p)^{Nt} S dt.$$

Another complication arises because we can sell s^* items at L^* and retain $s - s^*$ to sell at L . Suppose having a sale does not influence later purchases. Then we have

$$s^*(L^* + L^*pNs/S) \geq s^*(L + LpNs^*/2S)$$

and

$$s^*L^*(1+p)^{Ns^*/S} \geq L \int_0^{s^*/S} (1+p)^{Nt} S dt.$$

It can be shown that L^*/L is an increasing function of s^* — the less you sell, the greater markdown you can take and still break even. Now suppose people stock up at a sale so that no more are sold during the next s^*/S years. Then (1) and (2) simply have s replaced by s^* . In this case, L^*/L is a decreasing function of s^* and so there is no reason to retain any of the item. Unfortunately, the true situation usually lies between these two extremes.

CHAPTER 8

QUANTITATIVE DIFFERENTIAL EQUATIONS

8.1. ANALYTICAL METHODS

Pollution of the Great Lakes

In a recent article S. C. Chapra and A. Robertson (1977 Great Lakes eutrophication: The effect of point source control of total phosphorous. *Science* 196: 1448-1449) conclude that restricting phosphorous to one milligram per liter of sewage would significantly improve Lakes Michigan, Erie, and Ontario; however, agricultural runoff will also need to be reduced into western Erie.

PROBLEMS

2. (c) Assuming exactly 5 liters of blood gives 100 to 250 milligrams of drug; however assuming 4 to 6 liters (about 5 liters), gives 120 to 200 milligrams of drug. Let's allow a leeway of about 10%. Thus we aim for 130 to 180 milligrams. The time to fall from 180 to 130 is 3.75 hours. A reasonable time between doses is thus 4 hours. The required dose is 88 milligrams. Since dosage information is given in round numbers: 300 milligrams initially followed by 100 milligrams every 4 hours. This results in the drug being in the range 144 to 204 milligrams. This is a bit higher than we aimed for, but is probably acceptable. If, however, the 50% loss time is 10 hours instead of 8, we obtain a high of 247 milligrams. Fortunately, concentration is usually not as critical as in this problem.

3. We have

$$m_n'(t) = \frac{k}{2} \sum_{j=1}^{n-1} m_j(t)m_{n-j}(t) - \left(k \sum_{j=1}^{\infty} m_j(t)m_n(t) - \frac{k}{2} m_n^2(t) \right).$$

I don't know how to solve this equation; however, if we neglect the $\frac{km_n^2}{2}$ term, it can be solved: Set $s = \sum m_n$ and sum on n to obtain $s' = ks^2/2 - ks^2 = -ks^2/2$. Hence

$$s = \frac{2s_0}{2 + ks_0 t}.$$

The equation

$$m_n' = \frac{k}{2} \sum_{j=1}^{n-1} m_j m_{n-j} - k s m_n$$

can be solved inductively:

$$m_1(t) = 4s_0(2 + ks_0 t)^{-2}$$

and

$$m_n(t) = \frac{k}{2} (2 + ks_0 t)^{-2} \int_0^t \sum_{j=1}^{n-1} m_j(t)m_{n-j}(t) dt$$

for $n > 1$. I don't know how accurate this approximation is, but I suspect it is fairly good for large values of n provided s_0 is chosen carefully.

4. (c) Let $K = \int_r^s c(t) dt + \log \left(\frac{m(r)}{1 - m(r)} \right)$ where r and s are the age ranges. Then $m(s) = (1 + e^{-K})^{-1}$. If we take $s = \infty$, it is possible that $K \neq \infty$ and so $m(\infty) < 1$.
6. (a) If $p(t)$ is the proportion of consumers that buy our product and $A(t)$ is a measure of advertising, then

$$p' = -\lambda p + A(1 - p).$$

- (c) Uniform advertising at the level A_0 will lead to practically the steady state behavior and so

$$p = A_0 / (A_0 + \lambda).$$

An intensive campaign will raise p to the level p_0 and then $p(t) = p_0 e^{-\lambda t}$, which has an average value

$$\bar{p} = p_0(1 - e^{-6\lambda})/6\lambda$$

with time in months. To compare these we must estimate A_0 , p_0 , and λ .

7. (a) $\Delta d = (\alpha \Delta c)(1 - d)$. Thus $-\log(1 - d) = \alpha(c - c_0)$.

Fitting this to the data I obtained

	rotenone	pyrethrins	1:5	1:15
α	7.8	1.3	2.7	2.0
c_0	0.06	0.31	0.19	0.27

The fit is fairly good.

8.2. NUMERICAL METHODS

Towing a Water Skier

Of course, φ can be found easily enough once we have observed that $w = \cos \varphi$ and have computed w . Putting this aside, letting α be the angle of the boat's velocity vector and β the angle of the rope, we have $\varphi = \alpha - \beta$. We have $\tan \alpha = y'/x'$ and $\tan \beta = (y - s)/(x - r)$. Using

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

and rearranging we have

$$\tan \varphi = \frac{(x - r)y' - (y - s)x'}{(x - r)x' + (y - s)y'}$$

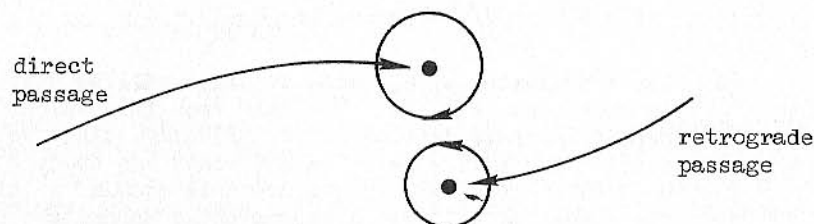
and so

$$\sin \varphi = \frac{(x - r)y' - (y - s)x'}{lv}$$

PROBLEMS

2. A. Toomre and J. Toomre proceeded as follows. The two galaxies are regarded as point masses. Thus the location of the galaxies is given analytically by the solution to the two body problem. To study distortion, each galaxy is surrounded by massless stars initially in circular orbits about the point mass. All the stars lie in the galactic

plane. Their positions are found by numerical integration. (It is a three body problem.) Slow passage (i.e., a parabolic orbit for the second galaxy relative to the first) with direct motion (see figure) produces the best tails, especially when the plane of the galaxy is not tilted too much with respect to the plane of its motion.



3. (a) A possible model is

$$S' = b - aIS$$

$$I' = aIS - cI$$

$$R' = cI.$$

- (b) Replace f' by $(f(t + \Delta t) - f(t))/\Delta t$ in (a).
 (c) The model in (a) oscillates if $ab < 4c^2$; however, the oscillations die out. (Section 9.2 is useful here.) The steady state ($S' = I' = 0$) is given by $S = c/a$, $I = b/c$. The period of oscillation is about

$$4\pi c / \sqrt{4abc^2 - a^2b^2}.$$

There are two problems (at least). First the oscillations are not self sustaining, which is discussed later. Second $(S + I + R)' = b$, so the population keeps growing. We remedy this. Let's just look at ages 5 through 9, ages of high susceptibility to measles. Then we use

$$S' = b - aIS - bS$$

$$I' = aIS - cI - bI$$

$$R' = cI - bI$$

$$S + I + R = 1.$$

The steady state solution is given by

$$S = (b+c)/a$$

$$I = b(a-b-c)/a(b+c).$$

Thus there will be a continuing infectious population if and only if $a > b + c$. In this case the period is about

$$P = 4(b+c)\pi/\sqrt{4b(b+c)^2(a-b-c) - a^2b^2}.$$

- (d) Let's evaluate a , b , and c . Time will be measured in weeks, so $P = 100$. The half week incubation period suggests setting $c = 2$. Finally, since we are considering a five year (= 250 week) age span, $b = 1/250$. Thus $a = 2.9$. Since this equals p times the number of children a random child encounters per week and the probability p is probably between 0.1 and 0.5, the value $a = 2.9$ looks reasonable. The equilibrium values are $S = (b+c)/a$ and $I = (a-b-c)b/a(b+c)$. Substituting in we obtain

$$S = 69\%, \quad I = 0.06\%, \quad \text{and} \quad R = 31\%.$$

This says that only about one child in two thousand is infectious in the equilibrium situation. Does this seem reasonable?

- (e) An equally good way to adjust a is by changing n . At any rate, let's suppose that there are two values a_w and a_s for winter and summer. If we assume that $b = 1/250$, $c = 2$, and the equilibrium values of I for winter and summer are in the ratio 1.6/0.4, then

$$(a_w - 2)/a_w = 4(a_s - 2)/a_s.$$

We need another equation. Since we had $a \approx 3$ before, let's set $a_w + a_s = 6$. Then $a_w = 3.7$, $a_s = 2.3$. It is interesting to solve the equations numerically for I , S , and R . It is also interesting to consider larger values of a_w and a_s — perhaps as large as 20.

CHAPTER 9

LOCAL STABILITY THEORY

9.2. DIFFERENTIAL EQUATIONS

Species Interaction and Population Size

As mentioned elsewhere, working with x'/x rather than x' is often quite useful. Since x'/x is independent of the units used to measure x , it is a more natural quantity than x' ; therefore, we can expect it to be easier to deal with in formulating a model. However at (and near) equilibrium values of x , $\partial x'/\partial u$ and $\partial(x'/x)/\partial u$ have the same sign for all u since

$$\frac{\partial x'}{\partial u} = x \frac{\partial(x'/x)}{\partial u} + \frac{x'}{x} \frac{\partial x}{\partial u} \approx x \frac{\partial(x'/x)}{\partial u}$$

since $x' = 0$ at equilibrium.

Keynesian Economics

What about government spending? If the amount is G and taxes are T , then the changes are

$$D = C + I + G$$

$$C = C(Y - T, R)$$

since $Y - T$ is disposable income. In a balanced budget, $T = G$. Setting $D = Y$ and $L = M$, computing partials with respect to G , and solving we obtain

$$Y_G = L_R(C_1 - 1)/\Delta$$

$$R_G = -L_Y(C_1 - 1)/\Delta$$

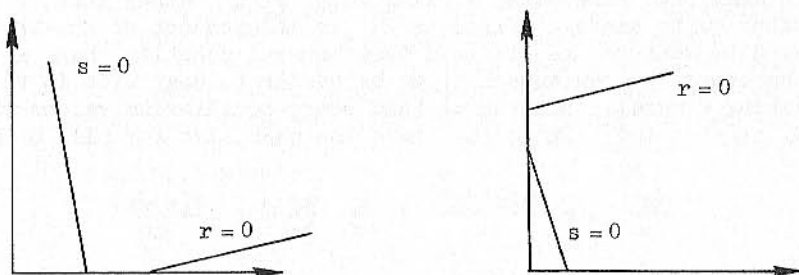
$$\Delta = (D_Y - 1)L_R - D_R L_Y$$

where C_1 is the partial of C with respect to disposable income. Both Y_G and R_G are positive; however the partial of disposable income with respect to G , $Y_G - 1$, is positive if

and only if $|L_{YR}| > |D_{RLY}|$. Since $Y_M > 0$ and $R_M < 0$, increasing government spending and the money supply together can lead to increased Y and unchanged R . If the government does not have a balanced budget, the situation is not in equilibrium. Our method cannot be used then.

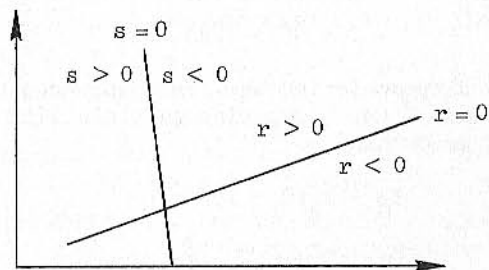
PROBLEMS

1. The $s=0$ curve is practically vertical. The $r=0$ curve has positive slope and is more nearly horizontal. The two situations shown here must be ruled out.



The first says predators can exist without prey. The second says that there can never be enough prey to support a predator population. If these are eliminated, there is a unique intersection.

2. (b) We have



Thus $s = \rho$ is to the left of $s=0$ and $r = \rho$ is above $r=0$.

- (c) Differentiate $r - \rho = 0$ and $s - \rho = 0$ and solve:

$$\frac{dx_0}{d\rho} = \frac{s_y - r_y}{\Delta} \frac{dy_0}{d\rho} = \frac{r_x - s_x}{\Delta}$$

where $\Delta = r_x s_y - r_y s_x$.

- (g) A model like that developed earlier in this problem can be used with ρ higher in peacetime (more fishing) than in wartime. One easily calculates $(x_0/y_0)' < 0$, which agrees with the data.

3. Symbiosis presents some problems. If we accept $|x_r x_x| > y_r y_y$ and $|y_s y_y| > x_s x_x$ as the text suggests, then

$$\left. \frac{dy}{dx} \right|_{r=0} > \frac{y}{x} > \left. \frac{dy}{dx} \right|_{s=0}$$

This implies that there is a unique intersection of $r=0$ and $s=0$, but it also implies that $r=0$ cannot intersect the y -axis. Hence species 1 can live without species 2, and vice versa. It is not clear why the model should make such a prediction. Actually it doesn't, because the above discussion tacitly assumes that $r=0$ does not contain the origin. In fact, it may do so and so may $s=0$. In this case $(0,0)$ is an unstable equilibrium.

5. (b) This probably requires some background in economics. Also, stability theory for two first order equations may be insufficient.
8. The stability for any number of equations follows from the fact that the eigenvalues of the matrix $\|\pi_i(a_i, a_j)\|$ have positive real part where $\pi_i > 0$ and a_i is a vector.
9. This can be done with a supply and demand type of a model with the money axis being "expected salary." Let W and Q stand for salary and number of people in the field. Then W' corresponds to demand and Q' to supply. Let $W' = D(W, Q)$ and $Q' = S(W, Q)$. Since stability requires $D_W + S_Q < 0$ and departments can control S , the prediction is quite simple: Keep graduate student production low when there are a lot of people in the field. Unfortunately, this does not allow for time delay.
10. See also Problem 8.2.3 and the discussion of it in this manual.

9.3. DIFFERENTIAL DIFFERENCE EQUATIONS

1. (a) Use the fact that the operators D and \Re , real part of, commute.

(b) Write $1 + \frac{i\omega e^{i\omega\tau}}{\lambda} = re^{i\theta}$. Then

$$v_n(t) - v_0(0) = \Re(Ae^{i(\omega t - n\theta)} r^{-n}).$$

Clearly $\limsup_{n \rightarrow \infty} |v_n(t) - v_0(0)| = \infty$ if and only if $r < 1$.

(c) This is simple algebra:

$$\left| 1 + \frac{i\omega e^{i\omega\tau}}{\lambda} \right|^2 = 1 + \frac{\omega^2}{\lambda^2} - \frac{2\omega \sin \omega\tau}{\lambda}.$$

2. Expanding $\mu_n(x'_{n-1} - x'_n)/(x_n - x_{n-1})$ about the equilibrium point and retaining only linear terms causes x_n and x_{n-1} to drop out since $x'_{n-1} = x'_n$ at equilibrium. Thus local stability is unchanged.

3. (a) A simple model is

$$x'(t) = -dx(t) + bx(t - \tau),$$

which leads to

$$be^{-\tau z} = d + z.$$

The root with largest real part is probably real.

(b) $x'(t) = bx(t) - bx(t - \lambda)$.

9.4. COMMENTS ON GLOBAL METHODS

PROBLEM

1. Suppose $s=0$ intersects the positive y axis at y_0 . Choose $y^* > y_0$. The existence of such a y_0 only requires that the environment cannot sustain a prey population above y_0 even in the absence of predators.

CHAPTER 10

STOCHASTIC MODELS

PROBLEMS

2. (b) Divide time into intervals of length t and set up Bernoulli trials with a success being no cars in an interval. Since a success guarantees a gap but not conversely, we obtain an overestimate for waiting time. The estimate is t/p . It seems difficult to decide how accurate the estimate is; however, one can compute the exact value: Divide time into intervals of length t/K and set up Bernoulli trials as before. A gap guarantees $K-1$ consecutive successes and is guaranteed by K consecutive successes. Letting $K \rightarrow \infty$ I obtained, after somewhat involved calculations, $(e^{\lambda t} - 1 - \lambda t)/\lambda$. Taking reasonable values such as $t = 20$ seconds and $\lambda = 3$ cars per minutes, $t/p = 54$ seconds. The correct value is 14 seconds.

(e) The simplest situation is to ignore the direction of motion of the vehicles. This can be done if both directions are Poisson and we require that no cars reach the crosswalk during the entire crossing period. Then λ is simply the total rate of traffic flow. Another simple situation occurs when there is a pedestrian island in the middle of the roadway. The waiting time is then the sum of the separate times. Now suppose only the half of the crosswalk that the pedestrian is in needs to be clear at the time. By shifting time for traffic on the far side by the time it takes the pedestrian to cross half the roadway, we can use the results from the first parts of the problem with λ equal to total flow rate and D equal to half the width of the roadway.

3. (f) This seems difficult to carry out.

Remark: Polymers are, of course, three dimensional; however, this adds no essential complication. Folding back on itself is a more serious problem, but neglecting that effect probably won't lead to large errors. Thus the root mean square length $(E(S^2))^{1/2}$ of a polymer

grows as the square root of the molecular weight of the polymer by (d). (For large n , $q^n \approx 0$.)

4. This problem could be assigned for class discussion. Afterwards groups could attack it in various ways suggested by the class discussion, some analytically and some by Monte Carlo simulation.
5. (c) The assumption that $y_0 - u_0$ is much larger than S allows us to neglect those beams which cool to a length less than u_0 .