

Math 636 Homework on chapter 2

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due 2008-09-17

Problem 6

(a)

If F is physical intensity and S is psychological intensity, we can relate F to S by analyzing either the functional relation $F(S)$ or its inverse $S(F)$.

Weber's Law states that for S_1 and S_2 differing by 1 jnd, the corresponding F_1 and F_2 differ by $kF(S)$, for some S between S_1 and S_2 . Making some approximations for small k and small $F_1 - F_2$ we can deduce from Weber's Law:

$$k \approx \frac{F_1 - F_2}{F_2} = \frac{F_1}{F_2} - 1 \approx \ln \left(1 + \frac{F_1}{F_2} - 1 \right) = \ln \left(\frac{F_1}{F_2} \right) = \ln(F_1) - \ln(F_2).$$

It is easier to derive $F(S)$ than $S(F)$. We can choose units so that in a neighborhood of interest a jnd in S is a difference of 1. Weber's Law then states that $F(S+1) - F(S) = kF(S)$, or else $kF(S+1)$, or something in between, in this neighborhood of interest. Fechner assumed that a jnd in S was always the same difference in S , so that we may extend the equation above to all values of S , and the k and the 1 remain constant.

The two difference equations just mentioned have solutions of

$$F = F(0)(1+k)^S$$

and

$$F = F(0)(1-k)^{-S},$$

respectively, which for small k are both approximately equal to

$$F = F(0)e^{Sk}.$$

Inverting this relation gives

$$S = \frac{1}{k} \ln \left(\frac{F}{F(0)} \right) = \frac{1}{k} \ln(F) + C.$$

Physically, one would expect $F = 0$ when $S = 0$, but allowing $F(0) = 0$ in the equations above would trash their usefulness. Weber's Law must be expected

to break down for small F , anyway, since no one can sense infinitesimal stimuli. The jnd just neighboring zero has to be a special case.

Loudness and brightness are measured in a logarithmic scale, because it is useful to deal with measured variables that are proportional to psychological intensity, instead of physical intensity.

(b)

Assume $\ln(F) = kS - C$. Then from the equal F ratios we can infer the following.

$$F_1/F_2 = F_3/F_4$$

$$\ln(F_1) - \ln(F_2) = \ln(F_3) - \ln(F_4)$$

$$kS_1 - kS_2 = kS_3 - kS_4$$

$$S_1 - S_2 = S_3 - S_4$$

This result could be inaccurate for large ΔS if differences of S are not quite proportional to counts of jnd steps between two values of S . This was Fechner's questionable assumption in our derivation in (a). Without some model for "noticing" we must allow that a "just noticeable difference" might be of any size in S . Inaccuracy of the law for extreme values may also be due to limitations in human perception.

(c)

After taking logarithms, examine any four points satisfying the conditions of Steven's law, so that

$$f_1 - f_2 = f_3 - f_4 = \delta$$

and

$$s_1 - s_2 = s_3 - s_4.$$

Assuming $s(f)$ is differentiable everywhere, the divided differences

$$\frac{s_1 - s_2}{f_1 - f_2} = \frac{s_3 - s_4}{f_3 - f_4}$$

will converge to a limit when s_2 and s_4 remain fixed and δ goes to zero. At those two limits, $s'(f_2) = s'(f_4)$. Since the points were chosen arbitrarily, $s'(f)$ is everywhere constant, say, equal to a . Integrating s' , we can write the first degree polynomial $s = af + \ln(b)$ for some positive constants, a and b . When we take exponentials of both sides to return to the original variables, this gives us $S = bF^a$.

(d)

Both laws can be approximately true for a range of F values because the two laws

$$S = bF^a$$

and

$$S = \frac{1}{k} \ln(F) + C$$

are approximately equal for a significant range of F , if constants are chosen suitably and $a < 1$.

Suitable constants for a region around $S_0 = S(F_0)$ can be chosen by matching function and derivative values. In this case:

$$a = \frac{1}{kS_0}$$

$$b = S_0 e^{C/S_0 - 1}.$$

The power law will not be concave down like the logarithm unless we choose the interpolation point $S_0 > \frac{1}{k}$.