Math 254 Practice Final

1. (10 pts) Find values of a and b such that the polynomial $p(x) = ax^4 + bx$ has a horizontal tangent at the point (1,3).

Answer: $a = -1, b = 4$

2. (20 pts) For which values of a and b does the following linear system

$$\begin{aligned} ax_1 + x_2 + x_3 &= 4 \\ x_1 + x_2 &= 6 \\ x_1 + x_3 &= b \end{aligned}$$

a. have one solution? Answer: $a \neq 2$
b. have no solutions? Answer: $a = 2, b \neq -2$
c. have infinitely many solutions? Answer: $a = 2, b = -2$

3. (10 pts) Find the following determinant

$$\begin{vmatrix} 5 & 31 & 0 & 0 & 8 \\ 2 & 8 & 0 & 0 & 7 \\ 75 & -87 & 1 & 86 & -21 \\ 0 & 1 & 0 & 0 & 0 \\ 38 & -47 & 0 & 1 & 92 \end{vmatrix}$$

Answer: 19

4. (20 pts) Find the inverses of the following matrices

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1.5 & -2 \\ -.5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 1.5 & -2 & 0 \\ -.5 & 1 & 0 \\ 0 & 0 & .2 \end{pmatrix}$$

5. (20 pts) A linear transformation $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ sends $\mathbf{e}_1$ to $2\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_2$ to $-3\mathbf{e}_1$, and $\mathbf{e}_3$

to $5\mathbf{e}_3$. What is the standard matrix of $\mathbf{T}$? What is the rank of $\mathbf{T}$?

Answer: $[T] = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. The rank of $[T]$ is 3.

6. (30 pts) Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 12 & 15 \\ 1 & 2 & 4 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 5 \\ 3 & 6 & 10 & 30 & 0 & 0 & 0 \\ 1 & 2 & 3 & 9 & 0 & 0 & 0 \end{pmatrix}$$

a. Find the rank and nullity of A and of A^T.
b. Find bases for the row space and column space of A.
c. Find bases for N=(null space of A) and for $N^\perp$.

Answer: The rref of A is $\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Thus rank$(A)$ =rank$(A^T) = 3$. By the dimension theorem, rank(A)+ nullity$(A) = 7$, thus nullity of A is 4. Similarly, rank(A^T)+ nullity$(A^T) = 5$, and thus nullity of A^T is 2.

A basis for the rowspace(A) is $\{(1,2,0,0,0,0,0),(0,0,1,3,0,0,0),(0,0,0,0,1,4,5)\}$.

A basis for the columnspace(A) is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \\ 10 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

A basis for the nullspace N of A is
$\{(-2,1,0,0,0,0,0),(0,0,-3,1,0,0,0),(0,0,0,0,-4,1,0),(0,0,0,0,-5,0,1)\}$.
Since N =rowspace$(A)^\perp$, $N^\perp$ is the same as (rowspace$(A)^\perp)^\perp$ =rowspace(A). Thus a basis for it is $\{(1,2,0,0,0,0,0),(0,0,1,3,0,0,0),(0,0,0,0,1,4,5)\}$.

7. (20 pts) (a) What is the transition matrix P from the basis

$$B' = \{\mathbf{u}_1' = 2\mathbf{e}_1 - \mathbf{e}_2, \mathbf{u}_2' = -3\mathbf{e}_1\}$$

to the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$?

(b) If $[u]_B = [1,2]$ and $[v]_{B'} = [3,4]$, find $[u]_{B'}$ and $[v]_B$.

Answer: $P = \begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}$. $[u]_{B'} = [-2, -5/3]$, and $[v]_B = [-6, -3]$.

8. (20 pts) (a) Find an orthonormal basis for $\mathbb{R}^2$ by applying the Gram-Schmidt orthonormalization process to the basis $\{(3,4),(1,1)\}$.

Answer: $v_1 = (3,4)$, and $u_1 = v_1/||v_1|| = (3/5, 4/5)$.

$v_2 = (1,1) - \langle (3/5,4/5),(1,1)\rangle\,(3/5,4/5) = (4/25, -3/25)$ and so $u_2 = v_2/||v_2|| = (4/5, -3/5)$.

(b) Find the inner product between the vectors $(3,4)$ and $(1,1)$ for the inner product generated by $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Answer:

$$\langle (3,4),(1,1)\rangle_{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)\cdot\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = (3,8)\cdot(1,2) = 19.$$

9. (20 pts) Find the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

and indicate the algebraic and geometric multiplicity for each eigenvalue you found.

Answer: $$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 3 & \lambda-3 \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -1 \\ 3 & \lambda-3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 0 \\ \lambda & -1 \end{vmatrix} = \lambda\,[\lambda(\lambda-3)+3)] - 1$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3$$

Thus the only eigenvalue is 1 and its algebraic multiplicity is 3. Its geometric multiplicity is

$$\text{nullity}\left(1\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}\right) = \text{nullity}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 3 & -2 \end{pmatrix}$$

Putting this into reduced row-echelon form, this equals

$$\text{nullity}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 3-\text{ rank}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 1.$$ Thus the geometric multiplicity is 1.

10. (30 pts) Find an orthogonal matrix P that diagonalizes $B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{pmatrix}$ and find $P^T BP$.

Answer: The characteristic polynomial is $\begin{vmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda-2 & -3 \\ 0 & -3 & \lambda-4 \end{vmatrix} = (\lambda - 6)\left[(\lambda-2)(\lambda-4)-9\right] = (\lambda-6)(\lambda-3+\sqrt{10})(\lambda-3-\sqrt{10})$.

To find each eigenspace, we calculate the nullspace of $(\lambda I - B)$. For $\lambda = 6$, we want the nullspace of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -3 & 2 \end{pmatrix}$, which is spanned by $(1,0,0)$. For $\lambda = 3+\sqrt{10}$, we want the nullspace of the matrix $\begin{pmatrix} -3+\sqrt{10} & 0 & 0 \\ 0 & 1+\sqrt{10} & -3 \\ 0 & -3 & -1+\sqrt{10} \end{pmatrix}$, which is spanned by $(0,3,1+\sqrt{10})$. A unit vector in this direction is $(0, \frac{3}{\sqrt{20+2\sqrt{10}}}, \frac{1+\sqrt{10}}{\sqrt{20+2\sqrt{10}}})$. Similarly, for $\lambda = 3-\sqrt{10}$ the easy eigenvector to write down is $(0,3,1-\sqrt{10})$ which, when normalized to unit length becomes $(0, \frac{3}{\sqrt{20-2\sqrt{10}}}, \frac{1-\sqrt{10}}{\sqrt{20-2\sqrt{10}}})$. Thus the orthogonal matrix that diagonalizes B is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{20+2\sqrt{10}}} & \frac{3}{\sqrt{20-2\sqrt{10}}} \\ 0 & \frac{1+\sqrt{10}}{\sqrt{20+2\sqrt{10}}}) & \frac{1-\sqrt{10}}{\sqrt{20-2\sqrt{10}}}) \end{pmatrix}$$

and $P^T BP = P^{-1}BP = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3+\sqrt{10} & 0 \\ 0 & 0 & 3-\sqrt{10} \end{pmatrix}$.

Part 2: Definitions

Give a short but correct definition of each of the following terms by completing the lines below:

A set S is *linearly independent* iff
Answer: None of its elements can be written as a non-trivial linear combination of its other elements.

A set $S \subset V$ *spans* V iff
Answer: Any element of V can be written as a linear combination of elements of S.

A set S is a *basis* of a vector space V iff
Answer: S is linearly independent and spans V.

The *dimension* of a vector space V is
Answer: The number of elements in any basis.

The *rank* of a matrix A is
Answer: The number of non-zero rows in its reduced row echelon form.

The *nullity* of a matrix A is
Answer: The dimension of its null space.

A number λ is an *eigenvalue* of the linear transformation A iff
Answer: There exists a vector v such that $Av = \lambda v$.

A matrix P *diagonalizes* the matrix A iff
Answer: $P^{-1}AP$ is a diagonal matrix.