13	06	04	23
6.3	6.4	6.5(A)	Total

1A

6.3

1) Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on R^2?

(a) $(0,1),(2,0)$

For set of vectors $(0,1),(2,0)$ we have

$(0,1)\cdot(2,0) = 0\cdot 2 + 1\cdot 0 = 0.$

$\therefore$ Vectors are orthogonal.

(b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} + \frac{1}{2} = 0.$

$\therefore$ Vectors are orthogonal.

(c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} - \frac{1}{2} = -1 \neq 0$

$\therefore$ Vectors are not orthogonal.

(d) $(0,0),(0,1)$.

$(0,0)\cdot(0,1) = 0\cdot 0 + 0\cdot 1 = 0$

$\therefore$ Vectors are orthogonal.

Orthogonal vectors : (a), (b), (d)
Non-orthogonal vectors : (c)

8) Verify that the set of vectors $\{(1,0),(0,1)\}$ is orthogonal with respect to the inner product $\langle u,v\rangle = 4u_1v_1 + u_2v_2$ on R^2 then convert it to an orthonormal set by normalizing the vectors.

Let's verify that given set of vectors is orthogonal or not.

For the inner product $\langle u,v\rangle = 4u_1v_1 + u_2v_2$ we have

$\langle(1,0),(0,1)\rangle = 4\cdot 1\cdot 0 + 1\cdot 0 = 0.$

So, the vectors are orthogonal.

The vector $(1,0)$ has a norm $\sqrt{\langle(1,0),(1,0)\rangle}$
$= \sqrt{4\cdot 1\cdot 1+0\cdot 0} = \sqrt{4} = 2$.

So the vector $\left(\frac{1}{2}, 0\right)$ has a norm 1.

The vector $(0,1)$ has a norm $\sqrt{\langle(0,1),(0,1)\rangle}$
$= \sqrt{4\cdot 0\cdot 0+1\cdot 1} = 1$.

So, the vector $(0,1)$ has a norm 1.

$\therefore \boxed{\text{Orthonormal set: } \left(\frac{1}{2}, 0\right), (0,1)}$

11) In each part, an orthonormal basis relative to the Euclidean inner product is given. Use Theorem 6.3.1 to find the coordinate vector of w with respect to the basis.

(a) $w = (3,7)$, $u_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\langle w, u_1\rangle = \frac{3}{\sqrt{2}} - \frac{7}{\sqrt{2}}$, $\langle w, u_2\rangle = \frac{3}{\sqrt{2}} + \frac{7}{\sqrt{2}}$

Therefore, if $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for an inner product space V, and u is any vector in V then $u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \cdots + \langle u, v_n\rangle v_r$

So, we have $w = -4\cdot\frac{1}{\sqrt{2}} u_1 + 10\cdot\frac{1}{\sqrt{2}} u_2$.

The coordinate vector of w relative to S is
$(w)_s = (\langle w, u_1\rangle, \langle w, u_2\rangle)$

$\therefore \boxed{(w)_s = \left(\frac{-4}{\sqrt{2}}, \frac{10}{\sqrt{2}}\right) = (-2\sqrt{2}, 5\sqrt{2})}$

(b) $w = (-1,0,2)$; $u_1 = (2/3, -2/3, 1/3)$;
$u_2 = (2/3, 1/3, -2/3)$, $u_2 = (1/3, 2/3, 2/3)$

$\langle w, u_1 \rangle = -\frac{2}{3} + \frac{2}{3}$

$\langle w, u_2 \rangle = -\frac{2}{3} - \frac{4}{3}$

$\langle w, u_3 \rangle = -\frac{1}{3} + \frac{4}{3}$

Therefore, if $S = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis for an inner product space V, and u is any vector in V then $u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \dots + \langle u, v_n \rangle v_n$.

So, we have $w = \frac{1}{3} 0 \cdot u_1 - \frac{1}{3} \cdot 6 \cdot u_2 + \frac{1}{3} \cdot 3 \cdot u_3$

$$\boxed{(w)_S = \left(\frac{1}{3} \cdot 0, -\frac{1}{3} \cdot 6, \frac{1}{3} \cdot 3\right) = (0, -2, 1)}$$

12) Let R^2 have the Euclidean inner product, and let $S = \{w_1, w_2\}$ be the orthonormal basis with $w_1 = (3/5, -4/5)$ $w_2 = (4/5, 3/5)$.

(a) Find the vectors u and v that have coordinate vectors $(u)_S = (1, 1)$ and $(v)_S = (-1, 4)$

If $S = \{w_1, w_2, \dots, w_n\}$ is an orthonormal basis for an inner product space V, and u is any vector V then

$u = \langle u, w_1 \rangle w_1 + \langle u, w_2 \rangle w_2 + \dots + \langle u, w_n \rangle w_n$

So, $u = 1 \cdot \left(\frac{3}{5}, -\frac{4}{5}\right) + 1 \cdot \left(\frac{4}{5}, \frac{3}{5}\right) = \left(\frac{7}{5}, -\frac{1}{5}\right)$

$v = -1\left(\frac{3}{5}, -\frac{4}{5}\right) + 4\left(\frac{4}{5}, \frac{3}{5}\right) = \left(\frac{13}{5}, \frac{16}{5}\right)$

$$\boxed{\begin{aligned} u &= (7/5, -1/5) \\ v &= (13/5, 16/5) \end{aligned}}$$

13) Let R^3 have the Euclidean inner product, and let $S = \{w_1, w_2, w_3\}$ be the orthonormal basis with $w_1 = (0, -3/5, 4/5)$, $w_2 = (1, 0, 0)$ and $w_3 = (0, 4/5, 3/5)$.

If $S = \{w_1, w_2, w_3\}$ is an orthonormal basis for an inner product space W, and u is any vector in W then

$u = \langle u, w_1\rangle w_1 + \langle u, w_2\rangle w_2 + \langle u, w_3\rangle w_3$

So, ~ yes

(9) Find the vectors u, v and w that have the coordinate vectors $(u)_S = (-2,1,2)$, $(v)_S = (3,0,-2)$, and $(w)_S = (5,-4,1)$.

$$u = -2\cdot\left(0,-\frac{3}{5},\frac{4}{5}\right) + 1\cdot(1,0,0) + 2\left(0,\frac{4}{5},\frac{3}{5}\right)$$

$$= (1, 14/5, -2/5)$$

$$v = 3\left(0,-\frac{3}{5},\frac{4}{5}\right) + 0\cdot(1,0,0) - 2\left(0,\frac{4}{5},\frac{3}{5}\right)$$

$$= (0, -17/5, 6/5)$$

$$w = 5\left(0,-\frac{3}{5},\frac{4}{5}\right) - 4(1,0,0) + 1\left(0,\frac{4}{5},\frac{3}{5}\right)$$

$$= \left(-4, -\frac{11}{5}, \frac{23}{5}\right)$$

$$\boxed{\begin{aligned} u &= (1, 14/5, -2/5) \\ v &= (0, -17/5, 6/5) \\ w &= (-4, -11/5, 23/5) \end{aligned}}$$

18) Let R^4 have the Euclidean inner product. Use the gram-schmidt process to transform the basis $\{u_1, u_2, u_3, u_4\}$ into an orthonormal basis.

$u_1 = (0,2,1,0)$, $u_2 = (1,-1,0,0)$, $u_3 = (1,2,0,-1)$, $u_4 = (1,0,0,1)$

$v_1 = u_1 = (0,2,1,0)$

$$v_2 = u_2 - \text{proj}_{w_1} u_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$= (1,-1,0,0) + \frac{2}{5}(0,2,1,0) = \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$v_3 = u_3 - \text{proj}_{w_2} u_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= (1,2,0,-1) - \frac{4}{5}(0,2,1,0) - \frac{3/5}{6/5}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$v_4 = u_4 - \text{proj}_{W_3} u_4$$

$$= u_4 - \frac{\langle u_4, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_4, v_2\rangle}{\|v_2\|^2} v_2 - \frac{\langle u_4, v_3\rangle}{\|v_3\|^2} v_3$$

$$= (1,0,0,1) - \frac{0}{5}(0,2,1,0) - \frac{6}{5}\left(1, -1/5, 2/5, 0\right) - \left(\frac{-1}{5}\right)\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= \left(4/15, 4/15, -8/15, 4/5\right)$$

The norm of these vectors are
$\|v_1\| = \sqrt{5}$, $\|v_2\| = \sqrt{6/5}$, $\|v_3\| = \sqrt{5/2}$, $\|v_4\| = 4/\sqrt{15}$

So, an orthonormal basis for R^4 is
$q_1 = v_1/\|v_1\|$, $q_2 = v_2/\|v_2\|$, $q_3 = v_3/\|v_3\|$, $q_4 = v_4/\|v_4\|$

$$\boxed{\begin{aligned} q_1 &= \frac{1}{\sqrt{5}}(0,2,1,0) \\ q_2 &= \sqrt{\frac{5}{6}}\left(1, \frac{-1}{5}, \frac{2}{5}, 0\right) \\ q_3 &= \sqrt{\frac{2}{5}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \\ q_4 &= \frac{\sqrt{15}}{4}\left(\frac{4}{15}, \frac{4}{15}, \frac{-8}{15}, \frac{4}{5}\right) \end{aligned}}$$

20) Let R^3 have the inner product $\langle u, v\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$. Use the Gram-Schmidt process to transform $u_1 = (1,1,1)$, $u_2 = (1,1,0)$, $u_3 = (1,0,0)$ into an orthonormal basis.

$$v_1 = u_1 = (1,1,1)$$

$$v_2 = u_2 - \text{proj}_{W_1} u_2$$

$$= u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$= (1,1,0) - \frac{2}{3}(1,1,1)$$

$$= \left(1/3, 1/3, -2/3\right)$$

$$v_3 = u_3 - \text{proj}_{W_2} u_3$$

$$= u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= (1,0,0) - \frac{1}{3}(1,1,1) - \frac{1}{2}(1,1,0)$$

$$= (1/6, -5/6, -1/3)$$

The norms of these vectors are
$\|v_1\| = \sqrt{3}$, $\|v_2\| = \sqrt{2/3}$, $\|v_3\| = \sqrt{6/5}$

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{2}{3}}\left(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \sqrt{\frac{5}{6}}\left(\frac{1}{6}, -\frac{5}{6}, \frac{-1}{3}\right)$$

$$\boxed{\begin{gathered} q_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad q_2 = \left(\frac{\sqrt{2}}{3\sqrt{3}}, \frac{\sqrt{2}}{3\sqrt{3}}, \frac{-2\sqrt{2}}{3\sqrt{3}}\right) \\ q_3 = \left(\frac{\sqrt{5}}{6\sqrt{6}}, \frac{-5\sqrt{5}}{6\sqrt{6}}, \frac{-\sqrt{5}}{3\sqrt{6}}\right) \end{gathered}}$$

21) The subspace of R^3 spanned by the vectors, $u_1 = (4/5, 0, -3/5)$ and $u_2 = (0,1,0)$ is a plane passing through the origin. Express $w = (1,2,3)$ in the form $w = w_1 + w_2$, where w_1 lies in the plane and w_2 is perpendicular to the plae

$$W = \text{Proj}_w W = (w, u_1) u_1 + (w, u_2) u_2$$

$$= \left(\frac{4}{5} - \frac{9}{5}\right) \cdot \left(\frac{4}{5}, 0, -\frac{3}{5}\right) + (2)(0,1,0)$$

$$= \left(-\frac{4}{5}, 0, \frac{3}{5}\right) + (0,2,0) = \left(-\frac{4}{5}, 2, \frac{3}{5}\right)$$

The component of w orthogonal to w is
$W_2 = \text{Proj}_{w^\perp} W = W - \text{Proj}_w W$

$= (1,2,3) - \left(-\frac{4}{5}, \frac{2}{0}, \frac{3}{5}\right)$
$= (9/5, 2, 12/5)$

$$\boxed{\begin{aligned} w_1 &= (-4/5, 20, 3/5) \\ w_2 &= (9/5, 0, 12/5) \end{aligned}}$$

23) Let R^4 have the Euclidean inner product. Express $w = (-1,2,6,0)$ in the form $w = w_1 + w_2$, where w_1 is in the space W spanned by $u_1 = (-1,0,1,2)$ and $u_2 = (0,1,0,1)$, and w_2 is orthogonal to W.

$$P = u_2 - \frac{u_1 \cdot u_2}{\|u_1\|^2} u_1$$

$$= (0,1,0,1) - \frac{2}{6}(-1,0,1,2)$$

$$\therefore P = (1/3, 1, -1/3, 1/3)$$

Now, $w_1 = \frac{w \cdot u_1}{\|u_1\|^2} u_1 + \frac{w \cdot P}{\|P\|^2} P$

$$= \frac{7}{6}(-1,0,1,2) + \frac{-1/3}{4/3}\left(\frac{1}{3}, 1, \frac{-1}{3}, \frac{1}{3}\right)$$

$$= \left(\frac{-7}{6}, 0, \frac{7}{6}, \frac{14}{6}\right) + \left(\frac{-1}{12}, \frac{-1}{4}, \frac{1}{12}, \frac{-1}{12}\right)$$

$$\therefore w_1 = \left(-\frac{5}{4}, \frac{-1}{4}, \frac{5}{4}, \frac{9}{4}\right)$$

Now as $w = w_1 + w_2$

$w_2 = w - w_1$

$= (-1,2,6,0) - (-5/4, -1/4, 5/4, 9/4)$

$= (-1/4, 9/4, 19/4, -9/4)$

$$\boxed{\begin{aligned} w_1 &= (-5/4, -1/4, 5/4, 9/4) \\ w_2 &= (-1/4, 9/4, 19/4, -9/4) \end{aligned}}$$

1.B

6.4

1) Find the normal system associated with the given linear system

(a) $$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

In this system the given linear system can be written as $Ax = b$. By defination, the normal system associated with $Ax = b$ is given as

$$A^T A x = A^T b.$$

where A^T denotes the matrix transpose obtained by exchanging A's rows and columns.

So, we have

$$A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}$$

Now, we are going to calculate the matrix $A^T A$ and the vector $A^T b$.

$$A^T A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$$

and

$$A^T B = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

Thus, the correct answer is

$$\boxed{\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}}$$

3) Find the last square solutions of the linear system $Ax = b$. and find the orthogonal projection of b onto the column space of A.

(a) $$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

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By defination, the normal system associated with $Ax = b$ is given as

$$A^T A x = A^T b.$$

$$A^T = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

$$\begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix} = 18 - 4 = 14 \neq 0$$

$$x_1 = \frac{\begin{vmatrix} 14 & -2 \\ -7 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix}} = \frac{70}{14} = 5$$

$$x_2 = \frac{\begin{vmatrix} 3 & 14 \\ -2 & -7 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix}} = \frac{7}{14} = \frac{1}{2}$$

Let W denote the column space of A and $\text{proj}_W b$ be the orthogonal projection of b on W.
If x is any solution of the normal system then

$$\text{proj}_W b = Ax.$$

$$\therefore \quad \text{proj}_W b = Ax = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -9/2 \\ -4 \end{bmatrix}$$

$$\boxed{\begin{array}{c} x_1 = 5, \quad x_2 = 1/2 \\ x = \frac{11}{2}, \quad y = \frac{-9}{2}, \quad z = -4 \end{array}}$$

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10) let W be the plane with equation $5x - 3y + z = 0$.

(a) Find a basis for W.

$$5x - 3y + z = 0$$
$$\therefore 5x = 3y - z$$
$$\Rightarrow x = \frac{3}{5}y - \frac{1}{5}z$$

It follows that y and z are free variables. Consider 2 vectors, V_1 and V_2 in W

$V_1 = (0.6, 1, 0)$ and $V_2 = (-1/5, 0, 1)$

where x components of V_1 and V_2 are defined from

$$x = \frac{3}{5}y - \frac{1}{5}z = 0.6y - 0.2z$$

let, V_1 and V_2 are linerly independent. (1)

If V_1 and V_2 are linerly dependent vectors then there are α and β such that $\alpha V_1 + \beta V_2 = 0$.

Writing the last eq^n by components gives

$$\alpha V_1 + \beta V_2 = (\alpha(0.6) - \beta(0.2), \alpha, \beta) = (0,0,0)$$

It follows that $\alpha = 0$ and $\beta = 0$.

Hence V_1 and V_2 are linerly depended, and these vectors form a basis for W.

$$\boxed{\begin{bmatrix} 0.6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.2 \\ 0 \\ 1 \end{bmatrix}}$$

(b) Use ~~the matrix obtained~~ Use formula (6) to find the standard matrix for the orthogonal projection of W.

$$5x - 3y + z = 0.$$
$$P = A(A^TA)^{-1}A^T$$

Since the coefficient of x is non zero, we can express x from the given equation. We have

$$x = \frac{3}{5}y - \frac{1}{5}z$$

It follows that y and z are free variables. Consider 2 vectors v_1 and v_2 in W

$v_1 = (3,5,0)$ and $v_2 = (-1,0,5)$ where x component of v_1 and v_2 are defined for $x = \frac{3}{5}y - \frac{1}{5}z$.

$$\therefore A = \begin{bmatrix} 3 & -1 \\ 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 5 & 0 \\ -1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 34 & -3 \\ -3 & 26 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \begin{bmatrix} 26 & 3 \\ 3 & 34 \end{bmatrix} = \frac{1}{875} \begin{bmatrix} 26 & 3 \\ 3 & 34 \end{bmatrix}$$

$$P = A \cdot (A^T A)^{-1} A^T = \begin{bmatrix} 3 & -1 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} \cdot \frac{1}{875} \begin{bmatrix} 26 & 3 \\ 3 & 34 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

$$= \frac{1}{875} \begin{bmatrix} 3 & -1 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 75 & 130 & 15 \\ -25 & 15 & 170 \end{bmatrix}$$

$$= \frac{1}{875} \begin{bmatrix} 250 & 375 & -125 \\ 375 & 650 & 75 \\ -125 & 75 & 850 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 10 & 15 & -5 \\ 15 & 26 & 3 \\ -5 & 3 & 34 \end{bmatrix}$$

$$\boxed{\text{std matrix} = \frac{1}{35} \begin{bmatrix} 10 & 15 & -5 \\ 15 & 26 & 3 \\ -5 & 3 & 34 \end{bmatrix}}$$

12) In R^3 consider the line l given by the equations $(x = t,\ y = t,\ z = t)$ and the line m given by the equation $(x = s,\ y = 2s - 1,\ z = 1)$. Let P be a point on l and let Q

be ~~ap~~ point on m. Find the values of ~~R~~ t and s that minimize the distance between the lines by minimizing the squared distance $\|P-Q\|^2$?

Let P and Q denote respectively the point on l that corresponds to t and the point on m that corresponds to s. Denote by u, the leading vector of the line l ~~by~~ and by v the leading vector of line m

$u = (1, 1, 1)$, $v = (1, 2, 1)$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} Ax &= b \\ \Rightarrow A^TAx &= A^Tb \end{aligned}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since $\begin{vmatrix} 3 & 4 \\ 4 & 6 \end{vmatrix} = 18 - 16 = 2 \neq 0$ it has only trivial sol.

Thus the target parameters t and s that minimize the distance between l and m are $t = 0$, $s = 0$.

$$\boxed{\begin{aligned} t &= 0 \\ s &= 0 \end{aligned}}$$

18) The relationship between the current I through a resistor and the voltage drop V across it is given by ohm's law $V = IR$. Successive experiments are performed in which a known current (measured in amps) is passed through a resistor of unknown resistance R and the voltage drop (measured in volts) is measured. This result in the (I, V) data (0.1, 1), (0.2, 2.1), (0.3, 2.9), (0.4, 4.2), (0.5, 5.1). The data

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is assumed to have measurement errors that prevent it from following Ohm's law precisely.

(C) Find the least square solutions of this system and interpret your result.

Let $\vec{I}$ denote the 5×1 matrix of the given system that can be considered as a vector in R^5; and $\vec{V}$ represent value.

$$\vec{I} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \end{bmatrix} \qquad \vec{V} = \begin{bmatrix} 1 \\ 2.1 \\ 2.9 \\ 4.2 \\ 5.1 \end{bmatrix}$$

From Ohm's law $\vec{I}\,x = \vec{V}$

$\therefore \vec{I}^T . \vec{I}\, x = \vec{I}^T . \vec{V}$

$$\therefore \vec{I}^T . \vec{I} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix} = \cancel{5.05}\ 0.55$$

$$\text{and } \vec{I}^T . \vec{V} = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2.1 \\ 2.9 \\ 4.2 \\ 5.1 \end{bmatrix} = 5.62$$

Thus, $0.55x = 5.62$

$$\therefore x = \frac{5.62}{0.55} = 10.21 \approx 10$$

Thus $R = 10$ is the best ~~exp~~ approximation in R^5 for the experiment data over all possible values of the coefficient of proportionality between the current I through a resistor and the voltage drop V across it.

$$\boxed{R = 10 \text{ ohms.}}$$

6.5 (A)

1) Find the coordinate vector for w relative to the basis $S = \{u_1, u_2\}$ for R^2.

(b) $u_1 = (2,-4)$, $u_2 = (3,8)$; $w = (1,1)$

The nonzero basis vectors for the new basis S, $u_1 = (2,1)$ and $u_2 = (3,8)$, are the coordinate vectors relative to the basis vectors $(1,0)$ and $(0,1)$ of the old basis A.

Then the transition matrix from the new basis S to the old basis is

$$P = [u_1 \mid u_2] = \begin{bmatrix} 2 & 3 \\ -4 & 8 \end{bmatrix}$$

$[W]_A = (1,1)$ is the coordinate vector for w relative to the old basis.

So, the coordinate vector for w relative to the new basis S will be

$$[W]_S = P^{-1}[W]_A = \frac{1}{28} \begin{bmatrix} 8 & -3 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{28}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5/28 \\ 6/28 \end{bmatrix}$$

$$\boxed{[W]_S = \begin{bmatrix} 5/28 \\ 6/28 \end{bmatrix}}$$

3) Find the coordinate vector for p relative to $S = \{P_1, P_2, P_3\}$.

(a) $p = 4 - 3x + x^2$, $P_1 = 1$, $P_2 = x$, $P_3 = x^2$

Suppose $S = \{P_1, P_2, P_3\}$ is a basis for a space P_2 of polynomials of degree at most 2, where

$P_1 = 1$, $P_2 = x$, $P_3 = x^2$

A polynomial $p = 4 - 3x + x^2$ in P_2 can be expressed in terms of the basis S as

$$p = 4P_1 - 3P_2 + P_3.$$

Thus numbers $4, -3$ and 1 are called the coordinates of p relative to S. The vector $(4,-3,1)$ constructed from

these coordinate vector P relative to S.
thus,

$$\boxed{[P]_S = (4,-3,1)}$$

4) Find the coordinate vector for A relative to $S = \{A_1, A_2, A_3, A_4\}$

$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

The vectors

$$V_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, V_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

in R^4 can be associated with the matrices A_1, A_2, A_3 & A_4.
The vectors V_1, V_2, V_3 and V_4 are linearly independent. Hence, these vectors can define a basis in R^4.

$$S' = \{V_1, V_2, V_3, V_4\}$$

The transition matrix from this new basis S' to the standard basis $B = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ in R^4 is

$$P = [V_1|V_2|V_3|V_4] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inverse matrix is

$$Q = P^{-1} = \begin{bmatrix} -1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, the matrix $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ can be represented by the vector $[V]_B = (2,0,-1,3)$ in R^4 whose coordinate vector relative to basis S' is

$$[v]_{B'} = Q\,[v]_B = \begin{bmatrix} -1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} = \overset{8B}{\begin{bmatrix} -1 \\ 1 \\ -1 \\ 3 \end{bmatrix}}$$

$$\therefore \boxed{\text{coordinate vector } v = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 3 \end{bmatrix}}$$

6) Consider the basis $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2\}$ for R^2, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

(a) Find the transition matrix from B' to B.

The transition matrix P from $B' = \{v_1, v_2\}$ to $B = \{u_1, u_2\}$ can be expressed in terms of it's column vector as

$$P = \big[\, [v_1]_B \,\big|\, [v_2]_B \,\big]$$

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

then, $[v_1]_B = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2u_1 + 1u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and $[v_2]_B = -3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 4 \end{bmatrix} = -3u_1 + u_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

and the transition matrix from the basis B' to the basis B is

$$P = \big[\, [v_1]_B \,\big|\, [v_2]_B \,\big] = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

$$\boxed{\text{Transition matrix } \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}}$$