6.5(B)	6.6	7.1	Total
/05	/07	/10	/22

1A

6.5 B:

6) Concider the basis $B = [u_1, u_2]$ and $B' = \{v_1, v_2\}$ for R^2 where

$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

(c) Compute the coordinate vector $[W]B$, where

$$W = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (9) to compute $[W]B'$

The transition matrix from $B' = \{v_1, v_2\}$ to $B = \{u_1, u_2\}$ can be expressed in terms of its column vector as

$$P = [[v_1]_B \mid [v_2]_B]$$

So, if

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

then $[v_1]_B \quad 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2u_1 + u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and $[v_2]_B = -3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3u_1 + 4u_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

and the transition matrix from the basis B' to the basis B is

$$P = [[v_1]_B \mid [v_2]_B] = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

The transition matrix from B to B' is

$$Q = P^{-1} = \frac{1}{8+3}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$

Now let's find the coordinate vectors of the vector $w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ relative to the basis B and B'

$$[W]_B = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3u_1 - 5u_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and $[W]_{B'} = Q[W]_B = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 \\ -13 \end{bmatrix}$

$$\boxed{\begin{aligned}[W]_B &= \begin{bmatrix} 3 \\ -5 \end{bmatrix} \\ [W]_{B'} &= \frac{1}{11}\begin{bmatrix} -3 \\ -13 \end{bmatrix}\end{aligned}}$$

8) Concider the bases $B = \{u_1, u_2, u_3\}$ and $B' = \{v_1, v_2, v_3\}$ for R^3 where

$$u_1 = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

(a) Find the transition matrix from B to B'

The matrix

$$Q_{B'} = P_{B'}^{-1} = ([v_1 | v_2 | v_3])^{-1} = \left(\begin{bmatrix} -6 & -2 & -2 \\ -6 & -6 & -3 \\ 0 & 4 & 7 \end{bmatrix}\right)^{-1}$$

$$= \frac{1}{144}\begin{bmatrix} -30 & 6 & -6 \\ 42 & -42 & 6 \\ -24 & 24 & 24 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} -5 & 1 & -1 \\ 7 & -7 & 1 \\ -4 & 4 & 4 \end{bmatrix}$$

$$[u_1]_{B'} = Q_{B'} u_1 = \frac{1}{24}\begin{bmatrix} -5 & 1 & -1 \\ 7 & -7 & 1 \\ -4 & 4 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 12 \\ -18 \\ 24 \end{bmatrix}$$

$$[u_2]_{B'} = Q_{B'} u_2 = \frac{1}{24}\begin{bmatrix} -5 & 1 & -1 \\ 7 & -7 & 1 \\ -4 & 4 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 18 \\ -36 \\ 16 \end{bmatrix}$$

$$[u_3]_{B'} = Q_{B'} u_3 = \frac{1}{24}\begin{bmatrix} 5 & 1 & -1 \\ 7 & -7 & 1 \\ -4 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 12 \\ -36 \\ 16 \end{bmatrix}$$

The transition matrix from B to B' is

$$Q = \left[[u_1]_{B'} \,\middle|\, [u_2]_{B'} \,\middle|\, [u_3]_{B'}\right] = \begin{bmatrix} 1/2 & 3/4 & 1/2 \\ -3/4 & -3/2 & -3/2 \\ 1 & 2/3 & 2/3 \end{bmatrix}$$

Transition matrix from B to B' : $\begin{bmatrix} \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{3}{4} & -\frac{3}{2} & -\frac{3}{2} \\ 1 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

(b) Compute the coordinate vector $[w]_B$ where

$$w = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

and use (9) to compute $[w]_{B'}$

The matrix

$$Q_B = P_B^{-1} = \left([u_1|u_2|u_3]\right)^{-1} = \left(\begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 6 \\ -3 & -1 & -1 \end{bmatrix}\right)^{-1}$$

$$= \frac{1}{48}\begin{bmatrix} 4 & -4 & -20 \\ -18 & 6 & 18 \\ 6 & 6 & -6 \end{bmatrix}$$

yields to the transition matrix from the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ to the basis B.

So,

$$[w]_B = \frac{1}{48}\begin{bmatrix} 4 & -4 & -20 \\ -18 & 6 & 18 \\ 6 & 6 & -6 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = \frac{1}{48}\begin{bmatrix} 48 \\ 48 \\ 48 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now we must find the coordinaties vectors for the basis vectors v_1, v_2, v_3 relative to basis B.

$$[v_1]_B = Q_B v_1 = \frac{1}{48}\begin{bmatrix} 4 & -4 & -20 \\ -18 & 6 & 18 \\ 6 & 6 & -6 \end{bmatrix}\begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix} = \frac{1}{48}\begin{bmatrix} 0 \\ 72 \\ -72 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/ \\ 3/ \end{bmatrix}$$

$$[v_2]_B = Q_B v_2 = \frac{1}{48}\begin{bmatrix} 4 & -4 & -20 \\ -18 & 6 & 18 \\ 6 & 6 & -6 \end{bmatrix}\begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix} = \frac{1}{48}\begin{bmatrix} -64 \\ 72 \\ -72 \end{bmatrix} = \begin{bmatrix} -4 \\ 3/ \\ -3/ \end{bmatrix}$$

$$[v_3]_B = Q_B v_3 = \frac{1}{48}\begin{bmatrix} 4 & -4 & -20 \\ -18 & 6 & 18 \\ 6 & 6 & -6 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \frac{1}{48}\begin{bmatrix} -136 \\ 144 \\ -72 \end{bmatrix} = \begin{bmatrix} 17/ \\ 3 \\ -3/ \end{bmatrix}$$

The transition matrix from B to B' is

$$Q = P^{-1} = \left([[v_1]_B \,|\, [v_2]_B \,|\, [v_3]_B] \right)^{-1} = \left(\frac{1}{48} \begin{bmatrix} 0 & -64 & -136 \\ 72 & 72 & 144 \\ -72 & -72 & -72 \end{bmatrix} \right)^{-1}$$

$$= \frac{1}{48 \times 331776} \begin{bmatrix} 5184 & 5184 & 576 \\ -5184 & -9792 & -9792 \\ 0 & 4608 & 4608 \end{bmatrix}$$

$$= \frac{1}{331776} \begin{bmatrix} 108 & 108 & 12 \\ -108 & -204 & -204 \\ 0 & 96 & 96 \end{bmatrix}$$

So,

$$[w]_{B'} = Q[w]_B = \frac{1}{331776} \begin{bmatrix} 108 & 108 & 12 \\ -108 & -204 & -204 \\ 0 & 96 & 96 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

$$= \frac{1}{331776} \begin{bmatrix} 264 \\ -72 \\ 288 \end{bmatrix}$$

$$\boxed{[w]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad [w]_{B'} = \frac{1}{331776} \begin{bmatrix} 264 \\ -72 \\ 288 \end{bmatrix}}$$

h) Consider the bases $B = \{p_1, p_2\}$ and $B' = \{q_1, q_2\}$ for P_1 where

$p_1 = 6 + 3x$, $p_2 = 10 + 2x$, $q_1 = 2$, $q_2 = 3 + 2x$

(b). Find the transition matrix from B to B'

The vector

$u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2)$, $v_1 = (c_1, d_1)$ and $v_2 = (c_2, d_2)$

can be associated with polynomials.

$p_1 = a_1x + b_1x$, $p_2 = a_2 + b_2x$, $q_1 = c_1 + d_1x$; $q_2 : c_2 + d_2x$

respectively.

$\therefore u_1 : (6,3)$, $u_2 : (10,2)$, $v_1 (2,0)$, $v_2 : (3,2)$.

Define the basis $S = \{u_1, u_2\}$ and $S' = \{v_1, v_2\}$ in R^2, which are actually the vector representations of the polynomial bases B and B' respectively.

The matrix,

$$Q_{S'} = P_S^{-1} = ([v_1|v_2])^{-1} = \left(\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\right)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$$

gives the transition matrix from the standard basis $\{(1,0),(0,1)\}$ to the basis S' (or from standard basis $\{1, x\}$ to the basis B' in polynomial notation.

Now we must find the coordinate vectors for the basis vectors u_1, u_2, relative to the basis S'.

$$[u_1]_{S'} : Q_{S'}u_1 = \frac{1}{4}\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 3/2 \end{bmatrix}$$

$$[u_2]_{S'} = Q_{S'}u_2 = \frac{1}{4}\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 14 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 1 \end{bmatrix}$$

So, the transition matrix from S to S' (and from B to B' respectively) is

$$Q = \left[[u_1]_{S'} \middle| [u_2]_{S'} \right] = \begin{bmatrix} 3/4 & 7/2 \\ 3/2 & 1 \end{bmatrix}$$

$\therefore$ $\boxed{\text{Transition matrix from } B \text{ to } B' : \begin{bmatrix} 3/4 & 7/2 \\ 3/2 & 1 \end{bmatrix}}$

(c)

Compute the coordinate vector $[P]_B$ where $p = -4 + x$ and use (a) to compute $[P]_{B'}$

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$$Q_S = P^{-1}S = \left([u_1 | u_2]\right)^{-1} = \left(\begin{bmatrix} 6 & 10 \\ 3 & 2 \end{bmatrix}\right)^{-1}$$

$$= \frac{1}{-18}\begin{bmatrix} 2 & -10 \\ -3 & 6 \end{bmatrix}$$

The polynomial $p = -4 + x$, represented by the vector $w = (-4, 1)$ in R^2, will have the following coordinate vector relative to the basis B.

$$[W]_S = Q_S W = \frac{-1}{18}\begin{bmatrix} 2 & -10 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \frac{-1}{18}\begin{bmatrix} -14 \\ 18 \end{bmatrix}$$

Now;

~~Now~~ $[V_1]_S = Q_S V_1 = \frac{-1}{18}\begin{bmatrix} 2 & -10 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{-1}{18}\begin{bmatrix} 4 \\ -6 \end{bmatrix}$

& $[V_2]_S = Q_S V_2 = \frac{-1}{18}\begin{bmatrix} 2 & -10 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{-1}{18}\begin{bmatrix} -14 \\ 3 \end{bmatrix}$

So, the transition matrix from S to S' (and from B to B' respectively) is

$$Q = P^{-1} = \left(\frac{-1}{18}\begin{bmatrix} 4 & -14 \\ -6 & 3 \end{bmatrix}\right)^{-1} = \frac{\cancel{1}}{\cancel{(-18)\cdot(-12)}} \cancel{\begin{bmatrix} 3 & 14 \\ 6 & 4 \end{bmatrix}}$$

$$= \left(\begin{bmatrix} \frac{2}{9} & -\frac{7}{9} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}\right)^{-1} = \frac{-9}{2}\begin{bmatrix} 1/6 & 7/9 \\ 1/3 & 2/9 \end{bmatrix}$$

So, $[P]_{B'} = [W]_{S'} = Q\,[W]_S = \frac{-9}{2}\begin{bmatrix} 1/6 & 7/9 \\ 1/3 & 2/9 \end{bmatrix}\begin{bmatrix} 7/9 \\ -1 \end{bmatrix}$

$$= \frac{-9}{2}\begin{bmatrix} -35/54 \\ 1/27 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 35 \\ -2 \end{bmatrix}$$

$$\therefore \boxed{[P]_B = \frac{-1}{18}\begin{bmatrix} -14 \\ 18 \end{bmatrix} \; ; \; [P]_{B'} = \frac{1}{12}\begin{bmatrix} 35 \\ -2 \end{bmatrix}}$$

6.6

(2) (b) Let $T: R^3 \longrightarrow R^3$ be multiplication by the matrix A ~~is~~ as shown below. Find $T(x)$ for the vector $x = (-2, 3, 5)$. Using the Euclidean inner product on R^3, verify that $\|T(x)\| = \|x\|$

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$$

$$T(x) = xA = (-2, 3, 5) \cdot \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix} = \left(\frac{-6}{3}, \frac{-15}{3}, \frac{9}{3}\right) = (-2, -5, 3)$$

$$\left.\begin{aligned} \|x\| &= \sqrt{(-2)^2 + (3)^2 + (5)^2} = \sqrt{4+9+25} = \sqrt{38} \\ \|T(x)\| &= \sqrt{(-2)^2 + (-5)^2 + (3)^2} = \sqrt{4+25+9} = \sqrt{38} \end{aligned}\right\} \text{same}$$

$$\boxed{\begin{aligned} T(x) &= (-2, -5, 3) \\ \|x\| &= \sqrt{38} \\ \|T(x)\| &= \sqrt{38} \end{aligned}}$$

6) Let a rectangular $x'y'$ coordinate system be obtained by rotating a rectangular xy coordinate system counterclockwise through the angle $\theta = 3\pi/4$.

(a) Find the $x'y'$-coordinates of the point whose x, y coordinates are $(-2, 6)$.

To find $x'y'$ coordinates of the point whose x, y coordinates are $(-2, 6)$ we use that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or $x' = x\cos\theta + y\sin\theta$

& $y' = -x\sin\theta + y\cos\theta$

$\therefore x' = (-2)\cos(3\pi/4) + (6)\sin(3\pi/4)$

& $y' = (2)\sin(3\pi/4) + (6)\cos(3\pi/4)$

$\therefore\ x' = (-2)(-1/\sqrt{2}) + (6)(1/\sqrt{2}) = \frac{2+6}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

$y' = (2)(1/\sqrt{2}) + (6)(-1/\sqrt{2}) = \frac{2-6}{\sqrt{2}} = \frac{-4}{\sqrt{2}} = -2\sqrt{2}$

$\therefore\ \boxed{(x', y') = (4\sqrt{2}, -2\sqrt{2})}$

8) Let a rectangular x', y', z' - coordinates ~~axe~~ system be obtained by rotating a rectangular x, y, z coordinate system counterclockwise about the z - axis (looking down z - axis) through the angle $\theta = \pi/4$.

(a) Find the x', y', z' - coordinates of the point whose x, y, z coordinates are $(-1, 2, 5)$.

To find x', y', z' coordinates of the point whose x, y, z coordinates are $(-1, 2, 5)$ we use that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or equivalently

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z$$

We have,

$$x' = (-1)\cos(\pi/4) + (2)\sin(\pi/4)$$
$$y' = (1)\sin(\pi/4) + (2)\cos(\pi/4)$$
$$z' = z$$

$\therefore$

$$x' = (-1)(1/\sqrt{2}) + (2)(\tfrac{1}{\sqrt{2}}) = 1/\sqrt{2}$$
$$y' = (1)(1/\sqrt{2}) + (2)(1/\sqrt{2}) = 3/\sqrt{2}$$
$$z' = 5$$

$\therefore\ \boxed{(x', y', z') = \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 5\right)}$

11) (a) A rectangular x', y', z' - coordinates system is obtained by rotating an xyz coordinate system counterclockwi about the y axis through an angle θ (looking along the positive y - axis toward the origin). Find a matrix A such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where (x, y, z) and (x', y', z') are the coordinates of the same point in the xyz - and $x'y'z'$ - systems respectively.

If we introduce unit vectors u_1, u_2, u_3 along the positive x, y, z axes, and unit vectors u_1', u_2' and u_3', $B' = \{u_1', u_2', u_3'\}$. Similarly for the rotation around the z-axis, we have

$$[u_1']_B = \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix} \quad \text{and} \quad [u_3']_B = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$$

moreover, since u_2' extends 1 unit up the positive y' - axis

$$[u_2']_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus the transition matrix from B' to B is

$$P = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

and the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Thus the new coordinates (x', y', z') of a point can be computed from its old coordinates (x, y, z) by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}$$

13) What conditions must a and b satisfy for the matrix $\begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$ to be orthogonal?

A square matrix A with the property $A^{-1} = A^T$ is said to be an orthogonal matrix.

We have $A^{-1} = \dfrac{1}{2(a^2+b^2)} \begin{bmatrix} b+a & a-b \\ b-a & a+b \end{bmatrix}$

and $A^T = \begin{bmatrix} a+b & a-b \\ b-a & b+a \end{bmatrix}$

Thus if $2(a^2+b^2) = 1$, $A^{-1} = A^T$ and so, the matrix A is orthogonal.

$$\boxed{a^2 + b^2 = 1/2}$$

16) Use the result in exercises 15 to determine whether multiplication by A is a rotation or a rotation followed by a reflection about the x-axis. Find the angle of rotation in either case.

(a) $A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

We use the fact that multiplication by A is a rotation followed by a reflection if $\det(A) = -1$.

we have

$$\det(A) = \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

since $\det(A) = 1$, multiplication by A is rotation

$$\boxed{\text{Rotation}}$$

(b) $A = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

$\det(A) = -\frac{1}{4} - \frac{3}{4} = -1$

since $\det(A) = 1$, multiplication by A is reflection

$\boxed{\text{Rotation}}$

7.1

1) Find the characteristic equation of the following matrices.

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

If A is $n \times n$ matrix then the characteristic equation of A is

$$\det(\lambda I - A) = 0$$

Here $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Now $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 \\ 8 & \lambda+1 \end{vmatrix}$

$\Rightarrow (\lambda-3)(\lambda+1) - 0 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$\therefore$ $\boxed{\text{characteristic equation: } \lambda^2 - 2\lambda - 3 = 0}$

4) Find the characteristic equation of the following matrices.

(a) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 0 & 1 \\ -2 & \lambda-1 & 0 \\ -2 & 0 & \lambda-1 \end{vmatrix}$

& characteristic equation: $\det(\lambda I - A) = 0$

$\Rightarrow (\lambda-4)\left[(\lambda-1)(\lambda-1)\right] + 1\left[0 - (-2)(\lambda-1)\right] = 0$

$\Rightarrow (\lambda-4)(\lambda^2 - 2\lambda + 1) + 2\lambda - 2 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\boxed{\begin{array}{l} \text{characteristic equation: } \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \\ \text{OR} \quad (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0 \end{array}}$

10) By the inspection find the eigenvalues of the following matrices.

(a) $\begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$

If A is $n \times n$ triangular (upper / lower / diagonal) matrix then the eigenvalues of A are the entries on the main diagonal of A.

$\therefore \boxed{\text{Eigen values} : \{-1, 5\}}$

(b) $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$

Given matrix is lower triangular matrix.

$\therefore \boxed{\text{Eigen values} : \{1, 3, 7\}}$

11) Find the eigen values of A^9 for

$$A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & 1/2 & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

We begin by finding the eigen values of A. A is upper triangular matrix. so, the given eigen values are the enteries on the main diagonal of the matrix.

$\therefore$ Eigen values for A = $1/2$, 1 & 2.

Let λ be an eigen vales of matrix B.

Then is $Bx = \lambda x$.

Now, $B^2x = B(Bx) = B(\lambda x) = \lambda Bx = \lambda(\lambda x) = \lambda^2 x$

$B^3x = B(B^2x) = \lambda^2 Bx = \lambda^3 x$.

$\vdots$

$B^n x = B(B^{n-1}x) = \lambda^{n-1} Bx = \lambda^n x$

where n is a whole and $n \geq 2$. Thus, if λ is an

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eigen values of B, then λ^n is an eigen values of B^n where n is a whole and $n \geq 2$

$\therefore$ Eigen values of A^9 = $(1/2)^9$, $(1)^9$ & $(2)^9$

= $\frac{1}{512}$, 1, 512

$\therefore$ $\boxed{\text{Eigen values of } A^9 = \frac{1}{512}, 1, 512}$

12) Find the eigenvalues and bases for the eigen spaces of A^{25} for

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

By defination, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigen ~~values~~ vector of A corresponding to λ if and only if x is a nontrivial solution of $(\lambda I - A)x = 0$.

We are given,

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

We begin by finding eigen values of A.

$\det(\lambda I - A) = \cancel{(\lambda+1)[0+1] + 2[0+1] + 2[-1+\lambda-2] = 0}$

$$\begin{vmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\therefore$ $(\lambda-1)^2(\lambda+1) = 0$

that is the eigen values of A are $\lambda = 1$ and $\lambda = -1$

If $\lambda = 1$ then $(\lambda I - A)x = 0$ beomes

$$\begin{bmatrix} 2 & -2 & -2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields

$$x_1 = -s - t, \quad x_2 = s, \quad x_3 = t$$

$$x = \begin{bmatrix} -t-s \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

since

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent, these vectors form a basis for the eigen space corresponding to $\underline{\lambda = 1}$

if $\lambda = -1$ the $(\lambda I - A)x = 0$ becomes

$$\begin{bmatrix} 0 & 2 & 2 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields

$$x_1 = 2s, \quad x_2 = -s \quad \text{and } x_3 = s$$

Thus, the eigenv vectors of A corresponding to $\lambda = -1$ are nonzero vectors of form

$$\begin{bmatrix} 2s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is a basis for the eigen space corresponding to $\lambda = -1$.

Now let λ be an eigenvalue of matrix B. That is $Bx = \lambda x$

Now $B^2x = B(Bx) = B(\lambda x) = \lambda(\lambda x) = \lambda^2 x$.

$B^3x = B(B^2x) = \lambda^2 Bx = \lambda^3 x$.

$\vdots$

$B^n x = B(B^{n-1}x) = \lambda^{n-1}Bx = \lambda^n x$.

where n is a whole and $n \geq 2$. Thus if λ is an eigenvalue of B, then λ^n is an eigenvalue of B^n, where n is a whole and $n \geq 2$.

If x is an eigenvector of B corresponding to the eigenvalue λ of matrix B, then x is an eigenvector of B^n corresponding to eigenvalue λ^n of matrix B^n, where n is a whole and $n \geq 2$.

That is, the eigenvalues of A^{25} are

$\lambda = (1)^{25} = 1$ and $\lambda = (-1)^{25} = -1$.

And the bases of the eigenspace are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$\boxed{\begin{array}{c} \lambda = -1, 1 \\ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \end{array}}$$

13) Let A be a 2×2 matrix, and call a line through the origin of R^2 invariant under A if Ax lies on the line when x does. Find equations for all lines in R^2, if any, that are invariant under the given matrix.

(a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

If A is $n \times n$ matrix and λ is a real number then, the following are equivalent.

(a) λ is an eigen value of A.

(b) the system of equations $(\lambda I - A)x = 0$ has non trivial solutions.

(c) There is a nonzero vector x in R^n such that $Ax = \lambda x$.

(d) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$

We are given $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

Note that Ax lies on the line when does if and only if $Ax = \lambda x$.

Hence, we should find the eigen values of A and the eigenspaces for each eigen value of A. These eigenspaces are lines invariant under the given matrix. We begin by finding eigen values.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 1 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda-4)(\lambda-1)+2 = 0$$

$\Rightarrow \quad \lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow$ ~~tat~~ $(\lambda-2)(\lambda-3) = 0$

that is, the eigen values are $\lambda = 2$ and $\lambda = 3$

By defination the eigen vectors corresponding to λ is a nontrivial solution of $(\lambda I - A)x = 0$.

<u>If $\lambda = 2$</u>, then $(\lambda I - A)x = 0$ beomes

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -2x + y = 0 \quad \Rightarrow \quad y = 2x.$

This is the line invariant under the given matrix

If <u>$\lambda = 3$</u>, then $(\lambda I - A)x = 0$ becomes

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x + y = 0 \quad \Rightarrow \quad y = x.$

This is the line invariant under the given matrix.

Thus we obtain two lines invariant under the given matrix, that is $y = 2x$ and $y = x$.

$$\boxed{\begin{aligned} y_1(x) &= x. \\ y_2(x) &= 2x \end{aligned}}$$

(c) $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \cancel{2-\lambda} & \cancel{3} \\ \cancel{0} & \end{vmatrix} \begin{vmatrix} \lambda-2 & -3 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-2)^2 = 0$$

$\therefore \lambda = 2$

If $\lambda = 2$ the $(\lambda I - A)x = 0$ beomes

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 3y = 0 \quad \Rightarrow \quad y = 0.$ (x-axis)

This is the only line invariant under the given matrix.

$$\boxed{y(x) = 0}$$

14) Find det(A) given that A has $p(\lambda)$ as it's characteristic polynomial.

(a) $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$

If A is nxn matrix, then the characteristic polynomial of A is

$$p(\lambda) = \det(\lambda I - A)$$

we are given $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5.$

That is $p(\lambda) = \det(\lambda I - A)$.

since the matrix is 3×3 $\det(\lambda I - A) = -\det(A - \lambda I)$

Hence $p(0) = -\det(A - 0I) = -\det(A)$

Now evaluate $p(0)$

so $p(0) = (0)^3 - 2(0)^2 + (0) + 5$

$\therefore p(0) = 5$

$\therefore \det(A) = -5.$

$$\boxed{\det(A) = -5}$$

(b) $p(\lambda) = \lambda^4 - \lambda^3 + 7.$

Here matrix is 4×4.

$\det(\lambda I - A) = \det(A - \lambda I)$

$\therefore \quad p(0) = \det(A - 0I) = \det(A)$

But $p(0) = (0)^4 - (0)^3 + 7$

$\therefore p(0) = 7$

$\therefore \boxed{\det(A) = 7}$