

Please write up solutions to the following problems on separate sheets of your own paper using one side only and labeling each sheet with your name, the problem number, and numbering the pages for each problem.

1. Prove directly from the  $\epsilon$ - $N$  definition of limit that  $\lim_{k \rightarrow \infty} (x_0 - 1/k)^3 = x_0^3$ .
2. Prove directly from the  $\epsilon$ - $\delta$  definition of limit that  $\lim_{x \rightarrow x_0} x^3 = x_0^3$ .
3. Prove directly from the  $\epsilon$ - $N$  definition of limit that  $\lim_{k \rightarrow \infty} (\frac{k^2}{k^2+1}) = 1$ .
4. Given that  $\lim_{x \rightarrow 0} \exp(x) = 1$ , prove that  $\lim_{x \rightarrow x_0} \exp(x) = \exp(x_0)$ .
- 5A. Consider the equivalence relation  $\simeq$  defined on  $\mathbb{R}^2 - \{(0, 0)\}$  as follows:

$$(x, y) \simeq (u, v) \text{ iff } \exists \lambda \in \mathbb{R} - \{0\} \text{ such that } u = \lambda x, v = \lambda y.$$

Show that  $\simeq$  defines an equivalence relation.

5B. The quotient set  $(\mathbb{R}^2 - \{(0, 0)\}) / \simeq$  is called the projective line. Show that each point of the projective line can be identified with a pair of diametrically opposed points on the unit circle.

6. Given  $p \in \text{boundary}(S)$ , show that there exist sequences  $x_n$  in  $S$  and  $y_n$  in the (complement of  $S$ ) =  $S^c = M - S$  such that  $x_n \rightarrow p$  and  $y_n \rightarrow p$ .
7. For each of the following sets, find the
  - interior
  - closure
  - boundary
  - set of accumulation points

- a.  $[3, 4] \cup ]4, 7]$
- b.  $\{x \mid x = 2p, p \in \mathbb{Z}\}$
- c. The rational numbers in  $]0, 2[$ .
- d.  $\{x \mid x = 4 + (-1)^n \binom{n+1}{n} \text{ for some } n \in \mathbb{Z}\} \cup \{3, 5\}$
- e.  $\{(x, y) \mid y = 1/x, 0 < x \leq 1\} \cup \{(0, 3)\}$
- f.  $\{(x, y) \mid 5 \geq x > 0, y = \sin(1/x)\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$

8. Assuming that  $f: \mathbb{R} \rightarrow \mathbb{R}$  represents an arbitrary continuous function, decide whether each of the following sets is necessarily

- open
- closed
- compact
- connected
- bounded

- a.  $f([a, b])$
- b.  $f^{-1}([a, b])$
- c.  $f(\{1, 3, -2.7\})$
- d.  $f^{-1}(\{1, 3, -2.7\})$
- e.  $f(\{x \mid x > 0\})$
- f.  $f^{-1}(\{x \mid x > 0\})$

9. Prove that the limit of a sum is the sum of the limits. More precisely:

9a. Show that if  $x_k \rightarrow L$  and  $y_k \rightarrow M$ , then  $(x_k + y_k) \rightarrow L + M$ .

and

9b. Show that if  $\lim_{x \rightarrow x_0} f(x) = L$  and  $\lim_{x \rightarrow x_0} g(x) = M$ , then  $\lim_{x \rightarrow x_0} (f(x) + g(x)) = L + M$ .

10. Given the fundamental theorem of calculus

$$\left( f \text{ continuous, } F(x) = \int_0^x f(t) dt \right) \Rightarrow (F'(x) = f(x)),$$

show that the mean value theorem for integrals

$$(f \text{ continuous}) \Rightarrow \left( \exists c \in ]a, b[ \text{ such that } \int_a^b f(t) dt = f(c)(b - a) \right),$$

follows from the mean value theorem for derivatives.