Math 534A Solutions to Practice Final

1. Prove directly from the ϵ-N definition of limit that $\lim_{k\to\infty}(x_0 - 1/k)^3 = x_0^3$.

Proof: Given $\epsilon > 0$, let $N = 3\lceil\frac{x_0^2+|x_0|+1/3}{\epsilon}\rceil$. Then $k > N$ implies

$$\epsilon > 3\frac{x_0^2 + |x_0| + 1/3}{k} > \frac{3|x_0|^2 + 3|x_0|/k + 1/k^2)}{k} > \frac{| - 3x_0^2 + 3x_0/k - 1/k^2|)}{k} = \left|(x_0 - 1/k)^3 - x_0^3\right|$$

2. Prove directly from the ϵ-δ definition of limit that $\lim_{x\to x_0} x^3 = x_0^3$.

Proof:
Given $\epsilon > 0$, let $\delta = \min\{1, \frac{\epsilon}{3(|x_0|+1)^2}\}$. Then $|x - x_0| < \delta$ implies $|x| < |x_0| + 1$ and so $|x^2 + xx_0 + x_0^2| < |x|^2 + |x||x_0| + |x_0^2| < (|x_0| + 1)^2 + (|x_0| + 1)|x_0| + |x_0|^2 < 3(|x_0| + 1)^2$. It then follows that $|x^3 - x_0^3| = |x - x_0||x^2 + xx_0 + x_0^2| < |x - x_0|3(|x_0| + 1)^2 < \delta 3(|x_0| + 1)^2 \leq \epsilon$.

3. Prove directly from the ϵ-N definition of limit that $\lim_{k\to\infty}(\frac{k^2}{k^2+1}) = 1$.

Proof:
Given $\epsilon > 0$, let $N = \lceil 1/\epsilon\rceil$. Then for $k > N$, we have $k^2 + 1 > k$ and so $k^2 + 1 > 1/\epsilon$. It then follows that $\left|\frac{k^2}{k^2+1} - 1\right| = \frac{1}{k^2+1} < \epsilon$.

4. Given that $\lim_{x\to 0}\exp(x) = 1$, prove that $\lim_{x\to x_0}\exp(x) = \exp(x_0)$.

Proof:
Since $\exp(x) = \exp(x-x_0)\cdot\exp(x_0)$, we can use a change of variable and the fact that limit of a constant equals the same constant to get $\lim_{x\to x_0}\exp(x) = \lim_{x\to x_0}(\exp(x-x_0)\cdot\exp(x_0)) = \lim_{x\to x_0}\exp(x-x_0)\cdot\lim_{x\to x_0}\exp(x_0) = \lim_{h\to 0}\exp(h)\cdot\lim_{x\to x_0}\exp(x_0) = 1\cdot\exp(x_0) = \exp(x_0)$.

5A. Consider the equivalence relation $\backsimeq$ defined on $\mathbb{R}^2 - \{(0,0)\}$ as follows:

$$(x, y) \backsimeq (u, v) \text{ iff } \exists\,\lambda \in \mathbb{R}-\{0\} \text{ such that } u = \lambda x, v = \lambda y.$$

Show that $\backsimeq$ defines an equivalence relation.

Proof:
The fact that $\backsimeq$ is reflexive follows by taking $\lambda = 1$. The fact that $\backsimeq$ is symmetric follows by taking $\lambda_2 = 1/\lambda$ since $((x, y) \backsimeq (u, v)) \Rightarrow (u = \lambda x, v = \lambda y) \Rightarrow (x = \lambda_2 u, y = \lambda_2 v) \Rightarrow ((u, v) \backsimeq (x, y))$. To prove transitivity, assume $(x, y) \backsimeq (u, v)$ and $(u, v) \backsimeq (a, b)$. Then there exist λ_1 and λ_2 such that $u = \lambda_1 x, v = \lambda_1 y, a = \lambda_2 u, b = \lambda_2 y$ and thus $a = \lambda_1\lambda_2 x$ and $b = \lambda_1\lambda_2 y$ implying $(x, y) \backsimeq (a, b)$.

5B. The quotient set $(\mathbb{R}^2 - \{(0,0)\})/\backsimeq$ is called the projective line. Show that each point of the projective line can be identified with a pair of diametrically opposed points on the unit circle.

Proof:
A point in the projective plane is an element of the quotient set and thus one equivalence class. Consider any $(x, y) \in (\mathbb{R}^2 - \{(0,0)\})$. Choosing $\lambda_1 = 1/(x^2+y^2)$ we see that $(x,y) \backsimeq (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$, and $\lambda_2 = -1/(x^2+y^2)$ gives $(x,y) \backsimeq (\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2})$ so the equivalence class contains two points on the unit circle equivalent to (x, y). Conversely, any point (u, v) on the unit circle such that $(u,v) \backsimeq (x,y)$ implies that $x = \lambda u$ and $y = \lambda v$ and thus $x^2+y^2 = \lambda^2(u^2+v^2) = \lambda^2$. Thus $\lambda = \pm\sqrt{(x^2+y^2)}$ and thus (u, v) was one of the two points $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ or $(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2})$. Thus any equivalence class contains exactly two points from the unit circle and these two points are diametrically opposed. It follows that we can identify (represent) each equivalence class with such a pair of diametrically opposed points that lie in that equivalence class.

6. Given $p \in$ boundary(S), show that there exist sequences x_n in S and y_n in the (complement of S) $= S^c = M - S$ such that $x_n \to p$ and $y_n \to p$.

Proof:
Since $p \in$ boundary(S), we know that for all $\epsilon > 0$, there exist points of S and points of S^c in $D(p, \epsilon)$. Taking a sequence of disks for smaller and smaller ϵ's, say $\epsilon = 1/n$, we can choose $x_n \in S \cap D(p, 1/n)$ and $y_n \in S^c \cap D(p, 1/n)$. It follows that x_n is in S and $x_n \to p$ and that y_n is in S^c and $y_n \to p$.

7. For each of the following sets, find the
· interior
· closure
· boundary
· set of accumulation points

- a. $[3, 4[\cup]4, 7]$

  interior$([3, 4[\cup]4, 7]) =]3, 4[\cup]4, 7[$.
  closure$([3, 4[\cup]4, 7]) = [3, 7]$.
  boundary$([3, 4[\cup]4, 7]) = \{3, 4, 7\}$.
  accumulation$([3, 4[\cup]4, 7]) = [3, 7]$.

- b. $\{x \mid x = 2p, p \in \mathbb{Z}\}$

  Let A be the set in this question. Then
  interior$(A) = \emptyset$.
  closure$(A) = A$.
  boundary$(A) = A$.
  accumulation$(A) = \emptyset$.

- c. The rational numbers in $]0, 2[$.

  interior$(\mathbb{Q}\cap]0, 2[) = \emptyset$.
  closure$(\mathbb{Q}\cap]0, 2[) = [0, 2]$.

boundary($\mathbb{Q} \cap]0,2[) = [0,2]$
accumulation($\mathbb{Q} \cap]0,2[) = [0,2]$.

- d. $\{x \mid x = 4 + (-1)^n(\frac{n+1}{n})$ for some $n \in \mathbb{Z}^+\} \cup \{3,5\}$

  Let A be the set in this question. Then
  interior$(A) = \emptyset$.
  closure$(A) = A$.
  boundary$(A) = A$
  accumulation$(A) = \{3,5\}$.

- e. $\{(x,y) \mid y = 1/x, 0 < x \leq 1\} \cup \{(0,3)\}$

  Let A be the set in this question. Then
  interior$(A) = \emptyset$.
  closure$(A) = A$.
  boundary$(A) = A$
  accumulation$(A) = \{(x,y) \mid y = 1/x, 0 < x \leq 1\}$.

- f. $\{(x,y) \mid 5 \geq x > 0, y = \sin(1/x)\} \cup \{(0,y) \mid -1 \leq y \leq 1\}$

  Let A be the set in this question. Then
  interior$(A) = \emptyset$.
  closure$(A) = A$.
  boundary$(A) = A$
  accumulation$(A) = A$.

8. Assuming that f: $\mathbb{R} \to \mathbb{R}$ represents an arbitrary continuous function, decide whether each of the following sets is necessarily
· open
· closed
· compact
· connected
· bounded

- a. $f([a,b])$

  not open
  necessarily closed
  necessarily compact
  necessarily connected
  necessarily bounded

- b. $f^{-1}([a,b])$

  not necessarily open
  necessarily closed
  not necessarily compact

not necessarily connected
not necessarily bounded

- c. $f(\{1, 3, -2.7\})$

  not open
  necessarily closed
  necessarily compact
  not necessarily connected
  necessarily bounded

- d. $f^{-1}(\{1, 3, -2.7\})$

  not necessarily open
  necessarily closed
  not necessarily compact
  not necessarily connected
  not necessarily bounded

- e. $f(\{x \,|\, x > 0\})$

  not necessarily open
  not necessarily closed
  not necessarily compact
  necessarily connected
  not necessarily bounded

- f. $f^{-1}(\{x \,|\, x > 0\})$

  necessarily open
  not necessarily closed
  not necessarily compact
  not necessarily connected
  not necessarily bounded

9. Prove that the limit of a sum is the sum of the limits. More precisely:

9a. Show that if $x_k \to L$ and $y_k \to M$, then $(x_k + y_k) \to L + M$.

Proof:
Since $x_k \to L$, given $\epsilon/2 > 0$, there exists N_1 such that $k > N_1 \Rightarrow |L - x_k| < \epsilon/2$. Similarly, since $y_k \to M$, given $\epsilon/2 > 0$, there exists N_2 such that $k > N_2 \Rightarrow |M - y_k| < \epsilon/2$. Then letting $N = \max\{N_1, N_2\}$ we have that for $k > N$

$$|(L + M) - (x_k + y_k)| < |L - x_k| + |M - y_k| < \epsilon.$$

9b. Show that if $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then $\lim_{x \to x_0}(f(x) + g(x)) = L + M$.

Proof:
Since $\lim_{x \to x_0} f(x) = L$, given $\epsilon/2 > 0$, there exists δ_1 such that $0 < |x - x_0| < \delta_1 \Rightarrow |L - f(x)| < \epsilon/2$. Similarly, since $\lim_{x \to x_0} g(x) = M$, given $\epsilon/2 > 0$, there exists δ_2 such that $0 < |x - x_0| < \delta_2 \Rightarrow |M - g(x)| < \epsilon/2$. Then letting $\delta = \min\{\delta_1, \delta_2\}$ we have that for $0 < |x - x_0| < \delta$,

$$|(L + M) - (f(x) + g(x))| < |L - f(x)| + |M - g(x)| < \epsilon.$$

10. Given the fundamental theorem of calculus

$$\left(f \text{ continuous, } F(x) = \int_0^x f(t)dt \right) \Rightarrow \left(F'(x) = f(x) \right),$$

show that the mean value theorem for integrals

$$(f \text{ continuous}) \Rightarrow \left(\exists\, c \in]a, b[\text{ such that } \int_a^b f(t)dt = f(c)(b - a) \right),$$

follows from the mean value theorem for derivatives.

Proof:
Since F is differentiable (presumably for all x), it is continuous on $[a, b]$ and differentiable on $]a, b[$. Applying the mean value theorem to F, we have that

$$F(b) - F(a) = \int_0^b f(t)dt - \int_0^a f(t)dt = \int_a^b f(t)dt = F'(c)(b - a) = f(c)(b - a)$$

for some $c \in]a, b[$.

Note: I should have given you also the additivity property of integrals over subintervals used in $\int_0^b f(t)dt - \int_0^a f(t)dt = \int_a^b f(t)dt$. Sorry.

Remarks: This test is clearly too long for a 2 hour exam but hopefully gave you a good review of the material. I promise to keep the length of the final reasonable.