Math 534A Solution to Problem 1.2.2 (and general hints about solving similar problems)

I am posting a solution to this problem for the following reasons:

The essential technique used here stays with us. Analysis is to a large extent the art of bounding things. This is the principal new manipulation you have to get good at, in order to succeed in the subject. The problem gives a convenient vehicle for added commentary describing the steps you have to go through to solve such problems.

The technique proceeds in stages which have acquired the names: back of the envelope portion and formal portion. The back of the envelope portion is so named because a formal proof never mentions it and it can be done on scratch paper. Don't be fooled. Some back of the envelope portions can take many large envelopes.

1 Back of the envelope portion (Part 1)

Analyze the behavior of the sequence. Play around. Evaluate a few terms. Can you compare to known sequences? The goal is to bound $|x_n - x|$ with an expression that decreases to zero as n increases, or to see that it is in fact not so bounded.

For the problem this is carried out by noticing the factorial which should suggest recursion. Next we notice that we can in fact write a recursion for the whole sequence as

$$x_{n+1} = \frac{3}{n+1} x_n.$$

Now a comparison to a geometric sequence is naturally suggested - except that the factor is not constant from one term to the next. This is OK provided we can find a constant that the factor is (at least eventually) less than. We note that the factor is $1/2$ once n reaches 5, so

$$x_n < (1/2)^{n-5} x_5 \quad \text{for} \quad n > 5.$$

Thus with the proviso that $n > 5$, we can choose the right hand side to equal ϵ.

2 Back of the envelope portion (Part 2)

Solve

$$\epsilon = (1/2)^{n-5} 3^5/5!$$

for ϵ, giving

$$n = 5 - \log_2(\frac{\epsilon 5!}{3^5})$$

Note that if ϵ is small, the argument of the log is less than one and the log is thus negative.

3 Formal Portion

We are then ready to write out the proof. Given $\epsilon > 0$, choose $N_0 = \max\{6, \lceil 5 - \log_2(\frac{\epsilon 5!}{3^5}) \rceil\}$. Then for any $n > N_0$, this gives us both inequalities

$$n > 5$$

and

$$n > 5 - \log_2(\frac{\epsilon 5!}{3^5})$$

Applying the 2^x function to both sides of the second inequality (note that the validity of this hinges on the monotonicity of 2^x at least for integer x), we get

$$2^{n-5} 3^5 / 5! < \epsilon.$$

But then noting the recursion

$$x_{n+1} = \frac{3}{n+1} x_n$$

and using the first inequality ($n > 6$) we have that

$$\frac{3}{n+1} < 1/2$$

and thus

$$x_n < 1/2 x_{n-1}$$

It then follows by induction that

$$x_n < (1/2)^{n-5} x_5$$

and thus

$$x_n < \epsilon$$

when $n > N_0$.