From W. Rudin, Principles of Mathematical Analysis

5.15. Theorem. *Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$, and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k. \tag{23}$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n. \tag{24}$$

For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n-1$; and (24) allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Proof: Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n \tag{25}$$

and put

$$g(t) = f(t) - P(t) - M(t-\alpha)^n \qquad (a \le t \le b). \tag{26}$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β. By (23) and (26),

$$g^{(n)}(t) = f^{(n)}(t) - n!M \qquad (a < t < b). \tag{27}$$

Hence the proof will be complete if we can show that $g^{(n)}(x) = 0$ for some x between α and β.

Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \ldots, n-1$, we have

$$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0. \tag{28}$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some x_1 between α and β, by the mean value theorem. Since $g'(\alpha) = 0$, we conclude similarly that $g''(x_2) = 0$ for some x_2 between α and x_1. After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1}, that is, between α and β.