Linear Algebra and Matrix Theory

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John Wiley and Sons, Inc.

New York and London

2 Finite Cones and Linear Inequalities

In this section and the following section we assume that F is the field of real numbers R, or a subfield of R.

If a set is closed under multiplication by non-negative scalars it is called a *cone*. This is in analogy with the familiar cones of elementary geometry with vertex at the origin which contain with any point not at the vertex all points on the same half-line from the vertex through the point. If the cone is also closed under addition it is called a *convex cone*. It is easily seen that a convex cone is a convex set.

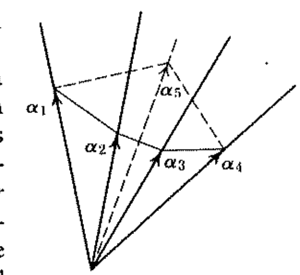

Fig. 3

If C is a convex cone and there exists a finite set of vectors $\{\alpha_1, \ldots, \alpha_p\}$ in C such that every vector in C can be represented as a linear combination of the α_i with non-negative coefficients, a *non-negative linear combination*, we call $\{\alpha_1, \ldots, \alpha_p\}$ the *generators* of C and call C a *finite cone*. The cone generated by a single non-zero vector is called a *half-line*. A dependable picture of a finite cone is formed by considering the half-lines formed by each of the generators as constituting an edge of a pointed cone as in Fig. 3. By considering a solid circular cone in R^3 it should be clear that there are convex cones that are not finite. A finite cone is the convex hull of a finite number of half-lines.

Let S be the largest subspace contained in C. If $S = \{0\}$, then S contains no line through the origin. In this case we say that C is *pointed*. If S is of dimension 1, then C is wedge shaped with S forming the edge of the wedge.

Given any subset $W \subset V$, let W^+ denote the set of all linear functionals that take on non-negative values for all $\alpha \in W$, that is, $W^+ = \{\phi \mid \phi\alpha \geq 0$ for all $\alpha \in W\}$. W^+ is closed under non-negative linear combinations and is a convex cone in $\hat{V}$. W^+ is called the *dual cone* or *polar cone* of W. Similarly, if $W \subset \hat{V}$, then W^+ is the set of all vectors which have non-negative values for all linear functionals in W. In this case, too, W^+ is called the dual cone of W. For the dual of the dual $(W^+)^+$ we write W^{++}.

Theorem 2.1. *(1) If* $W_1 \subset W_2$, *then* $W_1^+ \supset W_2^+$.

(2) $(W_1 + W_2)^+ = W_1^+ \cap W_2^+$ if $0 \in W_1 \cap W_2$.

(3) $W_1^+ + W_2^+ \subset (W_1 \cap W_2)^+$.

PROOF. (1) is obvious.

(2) If $\phi \in W_1^+ \cap W_2^+$, then for all $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ we have $\phi\alpha = \phi\alpha_1 + \phi\alpha_2 \geq 0$. Hence $W_1^+ \cap W_2^+ \subset (W_1 + W_2)^+$. On the other hand, $W_1 \subset W_1 + W_2$ so that $W_1^+ \supset (W_1 + W_2)^+$. Similarly $W_2^+ \supset (W_1 + W_2)^+$. Hence, $W_1^+ \cap W_2^+ \supset (W_1 + W_2)^+$. It follows then that $W_1^+ \cap W_2^+ = (W_1 + W_2)^+$.

(3) $W_1 \supset W_1 \cap W_2$ so that $W_1^+ \subset (W_1 \cap W_2)^+$. Similarly, $W_2^+ \subset (W_1 \cap W_2)^+$. It then follows that $W_1^+ + W_2^+ \subset (W_1 \cap W_2)^+$.

Theorem 2.2. *$W \subset W^{++}$ and $W^+ = W^{+++}$.*

PROOF. Let $W \subset V$. If $\alpha \in W$, then $\phi\alpha \geq 0$ for all $\phi \in W^+$. This means that $W \subset W^{++}$. It then follows that $W^+ \subset (W^+)^{++} = W^{+++}$. On the other hand from Proposition 2.1 we have $W^+ \supset (W^{++})^+ = W^{+++}$. Thus $W^+ = W^{+++}$. The situation is the same for $W \subset \hat{V}$.

A cone C is said to be *reflexive* if $C = C^{++}$.

Theorem 2.3. *A cone is reflexive if and only if it is the dual cone of a set in the dual space.*

PROOF. Suppose C is reflexive. Then $C = C^{++}$ is the dual cone of C^+. On the other hand, if C is the dual cone of $W \subset \hat{V}$, then $C = W^+ = W^{+++} = C^{++}$ and C is reflexive.

The dual cone of a finite cone is called a *polyhedral cone*. If C is a finite cone in $\hat{V}$ generated by the finite set $G = \{\phi_1, \ldots, \phi_q\}$, then $C^+ = D = \{\alpha \mid \phi_i\alpha \geq 0 \text{ for all } \phi_i \in G\}$. A dependable picture of a polyhedral cone can be formed by considering a finite cone, for we soon show that the two types of cones are equivalent. Each face of the cone is a part of one of the hyperplanes $\{\alpha \mid \phi_i\alpha = 0\}$, and the cone is on the positive side of each of these hyperplanes. In a finite cone the emphasis is on the edges as generating the cone; in a polyhedral cone the emphasis is on the faces as bounding the cone.

Theorem 2.4. *Let σ be a linear transformation of U into V. If C is a finite cone in U, then $\sigma(C)$ is a finite cone. If D is a polyhedral cone in V, then $\sigma^{-1}(D)$ is a polyhedral cone.*

PROOF. If $\{\alpha_1, \ldots, \alpha_p\}$ generates C, then $\{\sigma(\alpha_1), \ldots, \sigma(\alpha_p)\}$ generates $\sigma(C)$. Let D be a polyhedral cone dual to the finite cone E in $\hat{V}$. The following statements are equivalent: $\alpha \in \sigma^{-1}(D)$; $\sigma(\alpha) \in D$; $\psi\sigma(\alpha) \geq 0$ for all $\psi \in E$; $\hat{\sigma}(\psi)\alpha \geq 0$ for all $\psi \in E$; $\alpha \in (\hat{\sigma}(E))^+$. Thus $\sigma^{-1}(D)$ is dual to the finite cone $\hat{\sigma}(E)$ in $\hat{U}$ and is therefore polyhedral.

Theorem 2.5. *The sum of a finite number of finite cones is a finite cone and the intersection of a finite number of polyhedral cones is a polyhedral cone.*

PROOF. The first assertion of the theorem is obvious. Let $D_1, \ldots, D_r$ be polyhedral cones, and let $C_1, \ldots, C_r$ be the finite cones of which they are the duals. Then $C_1 + \cdots + C_r$ is a finite cone, and by Theorem 2.1 $D_1 \cap \cdots \cap D_r = C_1^+ \cap \cdots \cap C_r^+ = (C_1 + \cdots + C_r)^+$ is polyhedral.

Theorem 2.6. *Every finite cone is polyhedral.*

PROOF. The theorem is obviously true in a vector space of dimension 1. Let dim $V = n$ and assume the theorem is true in vector spaces of dimension less than n.

Let $A = \{\alpha_1, \ldots, \alpha_p\}$ be a finite set generating the finite cone C. We can assume that each $\alpha_k \neq 0$. For each α_k let W_k be a complementary subspace of $\langle\alpha_k\rangle$, that is, $V = W_k \oplus \langle\alpha_k\rangle$. Let π_k be the projection of V onto W_k along $\langle\alpha_k\rangle$. $\pi_k(C)$ is a finite cone in W_k. By the induction assumption it is polyhedral since dim $W_k = n - 1$. Then $\pi_k^{-1}(\pi_k(C)) = C_k$ is polyhedral by Theorem 2.4. Since $C \subset C_k$ for each k, C is contained in the polyhedral cone $C_1 \cap \cdots \cap C_p$.

We must now show that if $\alpha_0 \notin C$ then there is a C_j such that $\alpha_0 \notin C_j$. If not then suppose $\alpha_0 \in C_j$ for $j = 1, \ldots, p$. Then $\pi_j(\alpha_0) \in \pi_j(C)$ so that there is an $a_j \in F$ such that $\alpha_0 + a_j\alpha_j = \sum_{i=1}^p b_{ij}\alpha_i$ where $b_{ij} \geq 0$. We cannot obtain such an expression with $a_j \leq 0$ for then α_0 would be in C. But we can modify these expressions step by step and remove all the terms on the right sides.

Suppose $b_{ij} = 0$ for $i < k$ and $j = 1, \ldots, p$, that is, $\alpha_0 + a_j\alpha_j = \sum_{i=k}^p b_{ij}\alpha_i$. This is already true for $k = 1$. Then

$$\alpha_0 + (a_k - b_{kk})\alpha_k = \sum_{i=k+1}^p b_{ik}\alpha_i.$$

As before, we cannot have $a_k - b_{kk} \leq 0$. We could set $a_k - b_{kk} = a_k' > 0$ and proceed, but to avoid a change in notation we can assume $b_{kk} = 0$. Then

$$\left(1 + \frac{b_{kj}}{a_k}\right)\alpha_0 + a_j\alpha_j = \sum_{i=k+1}^{p} b_{ij}\alpha_i + \frac{b_{kj}}{a_k}(\alpha_0 + a_k\alpha_k)$$
$$= \sum_{i=k+1}^{p}\left(b_{ij} + \frac{b_{kj}}{a_k}b_{ik}\right)\alpha_i.$$

Upon division by $1 + \frac{b_{kj}}{a_k}$ we get an expression of the form

$$\alpha_0 + c_j\alpha_j = \sum_{i=k+1}^{p} d_{ij}\alpha_i, \quad j = 1, 2, \ldots, p,$$

with $c_j > 0$ and $d_{ij} \geq 0$. Continuing in this way we eventually get $\alpha_0 + c_j\alpha_j = 0$. This is impossible unless C is generated by a single vector α_1, and in this case C is easily shown to be polyhedral. Thus the assumption that $\alpha_0 \in C_j$ is untenable and $C = C_1 \cap \cdots \cap C_p$ is polyhedral.

Theorem 2.7. *A polyhedral cone is finite.*

PROOF. Let $C = D^+$ be a polyhedral cone dual to the finite cone D. We have just proven that a finite cone is polyhedral, so there is a finite cone E such that $D = E^+$. But then E is also polyhedral so that $E = E^{++} = D^+ = C$. Since E is finite, C is also.

Although polyhedral cones and finite cones are identical we retain both terms and use whichever is most suitable for the point of view we wish to emphasize. A large number of interesting and important results now follow very easily. The sum and intersection of finite cones are finite cones. The dual cone of a finite cone is finite. A finite cone is reflexive. Also, for finite cones part (3) of Theorem 2.1 can be improved. If C_1 and C_2 are finite cones, then $(C_1 \cap C_2)^+ = (C_1^{++} \cap C_2^{++})^+ = (C_1^+ + C_2^+)^{++} = C_1^+ + C_2^+$.

Our purpose in introducing this discussion of finite cones was to obtain some theorems about linear inequalities, so we now turn our attention to that subject. The following theorem is nothing but a paraphrase of the statement that a finite cone is reflexive.

Theorem 2.8. *Let*

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\geq 0 \\ &\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\geq 0 \end{aligned} \tag{2.1}$$

be a system of linear inequalities. If

$$a_1x_1 + \cdots + a_nx_n \geq 0$$

is a linear inequality which is satisfied whenever the system (2.1) *is satisfied, then there exist non-negative scalars* $(y_1, \ldots, y_m)$ *such that* $\sum_{i=1}^m y_i a_{ij} = a_j$ *for* $j = 1, \ldots, n$.

PROOF. Let ϕ_i be the linear functional represented by $[a_{i1} \cdots a_{in}]$, and let ϕ be the linear functional represented by $[a_1 \cdots a_n]$. If ξ represented by $(x_1, \ldots, x_n)$ satisfies the system (2.1), then ξ is in the cone C^+ dual to the finite cone C generated by $\{\phi_1, \ldots, \phi_m\}$. Since $\phi\xi \geq 0$ for all $\xi \in C^+$, $\phi \in C^{++} = C$. Thus there exist non-negative y_i such that $\phi = \sum_{i=1}^m y_i\phi_i$. The conclusion of the theorem then follows.

Theorem 2.9. *Let* $A = \{\alpha_1, \ldots, \alpha_n\}$ *be a basis of the vector space* U, *and let* P *be the finite cone generated by* A. *Let* σ *be a linear transformation of* U *into* V, *and let* β *be a given vector in* V. *Then one and only one of the following two alternatives holds: either*

(1) *there is a* $\xi \in P$ *such that* $\sigma(\xi) = \beta$, *or*

(2) *there is a* $\psi \in \hat{V}$ *such that* $\hat{\sigma}(\psi) \in P^+$ *and* $\psi\beta < 0$.

PROOF. Suppose (1) and (2) are satisfied at the same time. Then $0 > \psi\beta = \psi\sigma(\xi) = \hat{\sigma}(\psi)\xi \geq 0$, which is a contradiction.

On the other hand, suppose (1) is not satisfied. Since P is a finite cone, $\sigma(P)$ is a finite cone. The insolvability of (1) means that $\beta \notin \sigma(P)$. Since $\sigma(P)$ is also a polyhedral cone, there is a $\psi \in \hat{V}$ such that $\psi\beta < 0$ and $\psi\sigma(P) \geq 0$. But then $\hat{\sigma}(\psi)(P) \geq 0$ so that $\hat{\sigma}(\psi) \in P^+$.

It is apparent that the assumption that A is a basis of U is not used in the proof of this theorem. We wish, however, to translate this theorem into matrix notation. If ξ is represented by $X = (x_1, \ldots, x_n)$, then $\xi \in P$ if and only if each $x_i \geq 0$. To simplify notation we write "$X \geqq 0$" to mean each $x_i \geq 0$, and we refer to P as the *positive orthant*. Since the generators of P form a basis of U, the generators of P^+ are the elements of the dual basis $\hat{A} = \{\phi_1, \ldots, \phi_n\}$. It thus turns out that P^+ is the positive orthant of $\hat{U}$.

Let $\beta = \{\beta_1, \ldots, \beta_m\}$ be a basis of V and $\hat{B} = \{\hat{\beta}_1, \ldots, \hat{\beta}_m\}$ the dual basis in $\hat{V}$. Let $A = [a_{ij}]$ represent σ with respect to A and B, $B = (b_1, \ldots, b_m)$ represent β, and $Y = [y_1 \cdots y_m]$ represent ψ. Then $\hat{\sigma}(\psi)$ is represented by YA and $\hat{\sigma}(\psi) \in P^+$ if and only if $YA \geqq 0$. In this notation Theorem 2.9 becomes

Theorem 2.10. *One and only one of the following two alternatives holds: either*

(1) *there is an* $X \geqq 0$ *such that* $AX = B$, *or*

(2) *there is a* Y *such that* $YA \geqq 0$ *and* $YB < 0$.

Rather than continue to make these translations we adopt notational conventions which will make such translations more evident. We write $\xi \geqq 0$ to mean $\xi \in P$, $\xi \geqq \zeta$ to mean $\xi - \zeta \in P$, $\hat{\sigma}(\psi) \geqq 0$ to mean $\hat{\sigma}(\psi) \in P^+$, etc.

Theorem 2.11. *With the notation of Theorem 2.9, let* ϕ *be a linear functional in* $\hat{U}$, *let* g *be an arbitrary scalar, and assume* $\beta \in \sigma(P)$. *Then one and only one of the following two alternatives holds: either*

(1) *there is a* $\xi \geqq 0$ *such that* $\sigma(\xi) = \beta$ *and* $\phi\xi \geq g$, *or*

(2) *there is* $\psi \in V$ *such that* $\hat{\sigma}(\psi) \geqq \phi$ *and* $\psi\beta < g$.

PROOF. Suppose (1) and (2) are satisfied at the same time. Then $g > \psi\beta = \psi\sigma(\xi) = \hat{\sigma}(\psi)\xi \geq \phi\xi \geq g$, which is a contradiction.

On the other hand, suppose (2) is not satisfied. We wish to find a $\xi \in P$ satisfying the conditions $\sigma(\xi) = \beta$ and $\phi\xi \geq g$ at the same time. We have seen before that vectors and linear transformations can be used to express systems of equations as a single vector equation. A similar technique works here.

Let $U_1 = U \oplus F$ be the set of all pairs (ξ, x) where $\xi \in U$ and $x \in F$. U_1 is made into a vector space over F by defining vector addition and scalar multiplication according to the rules

$$(\xi_1, x_1) + (\xi_2, x_2) = (\xi_1 + \xi_2, x_1 + x_2),\ a(\xi, x) = (a\xi, ax).$$

Let $\tilde{P}$ be the set of all (ξ, x) where $\xi \in P$ and $x \geq 0$. It is easily seen that $\tilde{P}$ is a finite cone in U_1. In a similar way we construct the vector space $V_1 = V \oplus F$.

We then define Σ to be the mapping of U_1 into V_1 which maps (ξ, x) onto $\Sigma(\xi, x) = (\sigma(\xi), \phi\xi - x)$. It can be checked that Σ is linear. It is