

1. The sum  $S = - \sum_{i=1}^n p_i \ln(p_i)$  is a measure of the amount of randomness in a probability distribution  $P = (p_1, p_2, \dots, p_n)$ . Find the most random distribution over three categories, such that  $p_2 = 2p_3$ , i.e., find  $P = (p_1, p_2, p_3)$  which maximizes  $S$  subject to  $\sum_{i=1}^3 p_i = 1$  and  $p_2 = 2p_3$ .

2. Verify that the function

$$f_0(\mathbf{X}) = f_0(x, y, z) = x^2 + y^2 + z^2$$

subject to

$$f_1(\mathbf{X}) = f_1(x, y, z) = x^2 - 4y = 0$$

and

$$f_2(\mathbf{X}) = f_2(x, y, z) = x - y - 1 = 0$$

has a minimum at  $X_0 = (2, 1, 0)$  but that  $f_0'$  is not expressible as  $\lambda_1 f_1' + \lambda_2 f_2'$  at  $X_0$ . What happened?

3. A fund manager is choosing a portfolio of stocks in which to invest. The return per dollar investment in stock  $j$  has been on the average  $m_j$ , its variance has been  $s_{jj} = s_j^2$  and its covariance with stock  $k$  has been  $s_{jk}$ . The covariance matrix  $S = [s_{jk}]$  is symmetric and positive definite. Let  $x_j$  be the amount invested in stock  $j$  and let  $X = (x_1, \dots, x_n)$ . Then  $X$  has expected return  $m = \sum_j x_j m_j$  and variance  $s^2 = X^T S X$ .

- Formulate the general problem of maximizing the expected return on a dollar with a given variance or, equivalently, minimize the variance for a given mean return.
- What is the minimum variance portfolio which will guarantee a mean return of 0.13 per dollar invested in three stocks with the following data:

$$m_1 = 0.15, m_2 = 0.10, m_3 = 0.09,$$

$$S = \begin{bmatrix} 0.0625 & 0.0075 & -0.001 \\ 0.0075 & 0.0225 & 0.0010 \\ -0.001 & 0.0010 & 0.0012 \end{bmatrix}$$

4 A. Find  $(x, y, z)$  which maximizes  $f(x, y, z) = xyz$  subject to the constraint

$$c(x, y, z) = 2x + 3y + z = 1$$

where  $x$ ,  $y$ , and  $z$  are positive. Also find the value of the Lagrange multiplier.

Hint: form ratios.

4B. Find  $(x, y, z)$  which minimizes  $f(x, y, z) = 2x + 3y + z$  subject to the constraint

$$c(x, y, z) = xyz = 1/162$$

where  $x$ ,  $y$ , and  $z$  are positive. Also find the value of the Lagrange multiplier.

4C. Relate the two solutions in A and B above; pay special attention to the respective values of the Lagrange multipliers.